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Default and efficient debt markets

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Abstract

We examine the nature of debt contracts when repayment of debt cannot be fully enforced. We study outcomes in an infinite-horizon economy in which some individuals have access to a productive, intertemporal technology. Individuals without access to the technology must lend their savings to the rest. Borrowers can default on their debt at any time: lenders can capture a fraction of their investment incomes. Borrowers who default stand to lose the right to borrow in the future. These constitute the penalties of *capture* and *exclusion*.

We evaluate debt and repayment paths that can be sustained in this these penalties. The set of allocations that can be supported by default-free debt is fully characterized; this set is non-empty, convex, and contains a subset that is fully efficient.

We then evaluate the role of debt contracts in decentralizing constrained optima. Debt contracts that involve two-part pricing are shown to support efficient allocations subject to the no-default constraint. Efficiency is compatible with anonymous contracts.

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1. Introduction

We consider the extent of debt, and the nature of debt contracts, in a world where borrowers can default on debts. Our main interest lies in characterizing debt markets that support efficient investment and consumption paths.

The economy, and all participants, have an infinite-horizon. Individuals differ in access to an intertemporal production technology. The need for borrowing and lending arises from

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this: those who do not have direct access to production lend their savings to those who do. Borrowers can default their debts, and lenders cannot enforce full repayment directly. We assume that partial repayment can be directly enforced: lenders can capture a proportion λ of investment income. In addition, a borrower in default can be excluded from debt markets: he is then unable to borrow again. We evaluate the properties of consumption, investment, debt and repayment paths that can be sustained by the twin penalties of λ -capture and exclusion; characterize the set of efficient allocations that can be so sustained; and deduce the structure of one-period debt contracts that can support these constrained efficient allocations.

A debt plan, specifying a path of borrowing and repayments, can be sustained if borrowers do not default at any time and lenders do not capture their funds. For a debt plan to be default-free, the continued ability to borrow must generate a stream of rents for the borrower. For debt to be capture-free, lenders must earn enough from debt repayments. As default and capture are options available at every point of time, sustainable paths must promise sufficient future income to both sides at all times. Accordingly, sustainable debt plans must satisfy infinitely many inequality constraints. [Theorem 1](#) characterizes debt plans that can be sustained.

In [Theorem 2](#), we evaluate the implied restrictions on consumption allocations. The sequence of constraints on debt plans are equivalent to a single constraint on the path of aggregate consumption. The set of sustainable allocations is non-empty and convex. Any sustainable consumption allocation can be supported by at least one feasible and sustainable debt plan. This characterization is particularly useful in analyzing efficiency, as well as decentralization. We show, in [Theorem 3](#), that a subset of fully efficient, or first-best allocations, can be sustained; and that the set of efficient, sustainable allocations is non-empty for each λ , monotonically increasing in λ , and coincides with the set of fully efficient allocations when $\lambda = 1$. Hence, any efficient allocation can be sustained for some λ .

Turning to decentralization, we show, in [Theorem 4](#), that efficient and sustainable allocations can be achieved by trading in one-period debt contracts. These contracts involve *two-part pricing*. Lenders pay a fixed fee every period in order to participate in debt markets, and then earn a common marginal rate of return on their loans. At the margin, this rate of return equals the marginal productivity of capital. This last is, of course, a familiar characteristic of first-best paths. The fixed fee is paid to borrowers, and generates the rents that are necessary to prevent default. The marginal price is common to all debt contracts; the fixed fee is typically personalized. This is necessary if we want to implement all of the efficient solutions; an anonymous debt-contract achieves one of these efficient allocations. The allocation achieved by a competitive equilibrium can be reached if λ is large enough relative to the distribution of endowments. The associated debt contract involves a two-part tariff whenever $\lambda < 1$. Thus, in the absence of full enforcement, Walrasian allocations can be sustained only with non-Walrasian contracts.

The possibility that repeated trade may achieve superior outcomes in environments of limited enforcement has been a persistent theme in several papers, starting from [Allen \(1981, 1985\)](#), [Green \(1987\)](#), [Bulow and Rogoff \(1989\)](#). Many recent approaches, such as [Kimball \(1988\)](#), [Atkeson and Lucas \(1992\)](#), [Kehoe and Levine \(1993\)](#), [Coate and Ravallion \(1993\)](#), [Thomas and Worrall \(1994\)](#), [Kocherlakota \(1996\)](#), evaluate the role of exclusion as a threat in enforcing trade in exchange economies with individual endowment uncertainty. Typically, in these problems, individuals trade contracts, or securities for insurance purposes.

These contracts pay positive amounts when their income is low, and negative when income is high. Individuals who renege on these payments, and suffer the penalty of exclusion. They will do so unless the right to continue trade is sufficiently valuable: in an exchange economy, this simply requires that equilibrium utility levels are large enough relative to autarky.

This approach runs into some difficulties in situations where participants are able to save, independently of access to markets. An individual with access to storage can plan to default on their payments; excluded from asset markets, they can consume their savings for the rest of their lives. This possibility raises the incentive to default, and lowers the deterrent effect of exclusion. Typically, this reduces the set of outcomes which can be sustained by repeated trading with the threat of exclusion. Emphasized first by Allen (1984), this insight was used to derive a significant impossibility result by Bulow and Rogoff (1989). With complete, competitive markets, the threat of exclusion cannot deter default by sovereign countries: sovereignty implies $\lambda = 0$ in our terminology. Storage, or investment, is central to our problem. Individuals who have access to storage are precisely the ones who are likely to default. The right to borrow is valuable only if it commands rents. The penalty of exclusion has a deterrent effect if these rents are sufficiently high. We simplify the problem in other respects by assuming that there is no uncertainty in production or endowments. The negative result of Bulow and Rogoff—that borrowing and lending cannot be sustained if $\lambda = 0$ —continues to be true. For positive capture rates, some amount of default-free debt can be sustained.

Section 2 sets out the model. In Section 3, we demonstrate efficient allocations in a setting of full, and costless enforcement. The possibility of default, and the effect of penalties, are set out in Section 4: the two penalties impose incentive-compatibility constraints on the entire path of debt. Debt plans, and consumption allocations that can be sustained by the threat of exclusion are fully characterized in Section 5; in Section 6, we find constrained efficient allocations, and show that a subset of fully efficient paths are sustainable. In Section 7, we turn to decentralization of these constrained efficient allocations. Specifically, we propose debt contracts which require two-part pricing, as a participation fee plus return on loans, and show that every constrained efficient allocation path can be achieved as an equilibrium in an economy with two-part debt contracts, and redistribution of endowments. Section 7 concludes.

2. The economy

We consider an economy with a continuum of infinitely-lived individuals, indexed by $h \in [0, 1]$. Time is discrete, $t = 0, 1, 2, \dots$. There is one aggregate commodity each period, which can be consumed or invested. Individual preferences are represented by

$$\sum_{t=0}^{\infty} \beta^t U(c_{ht})$$

where $c_{ht} \geq 0$ is the consumption of individual h at time t , and $\beta \in (0, 1)$ their common discount factor. Individuals differ in endowments and in access to productive technologies.

The counting measure is assumed uniform on $[0,1]$, i.e. $d(h) = dh$ for each $0 \leq h \leq 1$. We make several simplifying assumptions.

Assumption 1 (Preferences). The period utility function is

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}; \quad \sigma > 0.$$

Assumption 2 (Endowments). Individual h has endowment $e_{h0} > 0$ in the initial period, and 0 thereafter. The aggregate endowment at $t = 0$ is $e_0 = \int_0^1 e_{h0} dh$.

Assumption 3 (Technology). The economy has a production technology that yields output y_t according to

$$y_{t+1} = (1 + \rho)k_t; \quad k_t \geq 0$$

where k_t is the amount of capital invested at t . Capital depreciates fully on use. The parameter $\rho \geq 0$ satisfies the restriction

$$\beta(1 + \rho)^{1-\sigma} < 1.$$

These assumptions imply that an efficient path exists. This path displays sustained growth in consumption whenever $\beta(1 + \rho) > 1$, which is compatible with [Assumption 3](#). The assumption is familiar from the analysis of convex models of growth (e.g. [Jones and Manuelli \(1990\)](#)). This version is often called the $A-k$ model.

The next assumption is crucial for our analysis.

Assumption 4 (Access). Let $0 < \pi < 1$. Individual h has direct access to the production technology if, and only if $h \geq \pi$.

Individuals who have direct access to the intertemporal technology can use this to transfer income and consumption over time. The rest can do so only by lending to them. Define $e_L = \int_0^\pi e_h dh$, the total endowment of potential lenders.

Finally, we define concepts used repeatedly in our analysis. We write

$$X_t = \{x_t, x_{t+1}, x_{t+2}, \dots\}$$

for an infinite sequence starting from t ; obviously, X_0 describes an entire path. The *value* of this sequence, discounted at rate ρ , is

$$X_t^+ = \sum_{i=0}^{\infty} \frac{x_{t+i}}{(1 + \rho)^i}.$$

A *consumption plan* for individual h is

$$C_{h0} = \{c_{h0}, \dots, c_{ht}, \dots\}; \quad c_{ht} \geq 0.$$

An *allocation* is a collection of consumption plans for all individuals

$$C_0 = [C_{h0}; 0 \leq h \leq 1].$$

Similarly, an *investment plan* is

$$K_{h0} = \{k_{h0}, \dots, k_{ht}, \dots\}; \quad k_{h0} \geq 0,$$

where k_{ht} is investment in individual h 's technology. An *investment allocation* satisfies the access restriction $\int_0^\pi k_{ht} dh = 0$, and is defined by

$$\mathcal{K}_0 = [k_{j0}; \pi \leq j \leq 1].$$

Let $c_t = \int_0^1 c_{ht} dh$ and $k_t = \int_\pi^1 k_{ht} dh$ be aggregate consumption and investment levels at t . An allocation \mathcal{C}_0 generates an aggregate consumption plan $C_0 = \{\dots, c_t, \dots\}$; similarly, an investment allocation generates an aggregate investment plan $K_0 = \{\dots, k_t, \dots\}$. An individual consumption plan C_{h0} has value C_{h0}^+ , and generates utility

$$V(C_{h0}) = \sum_{t=1}^\infty \beta^t U(c_{ht}).$$

We note that valuation, and preferences are both recursive: a continuation plan C_{ht} , evaluated from period t , is of value C_{ht}^+ and yields utility $V(C_{ht})$.

Finally, we write

$$e_L = \int_0^\pi e_h dh; \quad e_H = \int_\pi^1 e_h dh; \quad c_{Lt} = \int_0^\pi c_{ht} dh; \quad c_{Ht} = \int_\pi^1 c_{ht} dh$$

for endowments and consumption levels of the types of individuals in the economy.

3. Efficient allocations

An allocation $\mathcal{C}_0 = [C_{h0}; 0 \leq h \leq 1]$ is *feasible* if it can be generated by an investment allocation satisfying access restrictions, such that

$$\int_0^1 [c_{h0} - e_{h0}] dh + \int_\pi^1 k_{j0} dh \leq 0;$$

$$\int_0^1 c_{ht} dh + \int_\pi^1 [k_{jt} - (1 + \rho)k_{j,t-1}] dh \leq 0; \quad k_{jt} \geq 0 \text{ for } \pi \leq j \leq 1.$$

Most aspects of the model are well understood. We want to draw attention to three properties that are important in our analysis.

Property 1 (Feasibility). *Let \mathcal{C}_0 be an allocation, and C_0 the associated aggregate consumption plan. \mathcal{C}_0 is feasible if, and only if, C_0 satisfies*

$$C_0^+ \equiv \sum_{t=0}^\infty \frac{c_t}{(1 + \rho)^t} \leq e_0 \quad (\mathbf{F}).$$

This condition obtains by recursing the constraints $c_t + k_t \leq (1 + \rho)k_{t-1}$, and verifying that $c_t + k_t \geq 0$ whenever it is satisfied. This representation is particularly useful in obtaining efficient paths. The utility of C_{h0} is $V(C_{h0})$. An efficient allocation solves a program

$$\max \int_0^1 V(C_{h0}) \omega(h) dh \quad (\mathbf{P}_*)$$

subject to (\mathbf{F}) , for some weight function $\omega(h)$ satisfying $\omega(h) \geq 0$ and $\int_0^1 \omega(h) dh = 1$.

Property 2 (Efficient paths). *Let $1 + \theta = [\beta(1 + \rho)]^{1/\sigma}$. Along any efficient path, aggregate consumption is*

$$c_0^* = e_0 \frac{\rho - \theta}{1 + \rho}; \quad c_t^* = c_0^* (1 + \theta)^t.$$

Individual consumption levels are $c_{ht}^ = \gamma(h)c_t^*$, where $\gamma(h) = \omega(h)^{1/\sigma} / \int_0^1 \omega(h)^{1/\sigma} dh$.*

We note that [Assumption 3](#) implies $\theta < \rho$; and that $\theta > 0$ whenever $\beta(1 + \rho) > 1$. The efficient path of aggregate consumption is independent of welfare weights $\omega(h)$. Individuals consume a stationary fraction of aggregate consumption. This is, of course, a consequence of the fact that the period utility functions are homothetic. Along the optimal path, aggregate levels of investment and output are

$$k_t^* = (1 + \theta)^{t+1} \frac{e_0}{1 + \rho}$$

$$y_t^* = (1 + \theta)^t e_0.$$

Aggregate savings are invested in some or all of the technologies $h \geq \pi$. With constant returns to scale, and no capacity constraints, it is a matter of indifference whether some of these are kept idle. However, because of [Assumption 4](#), we know that the efficient path requires non-trivial transfers between individuals with and without access, which we can interpret as borrowing and lending.

Property 3 (Efficient debt). *Let C_0^* be an efficient allocation; and $\gamma_L = c_{Lt}^* / c_t^* = \int_0^\pi \gamma(h) dh$. An efficient path implies borrowing and lending whenever $\gamma_L > 0$. Aggregate debt levels are*

$$d_t^* = \gamma_L k_t^* = \gamma_L \frac{1 + \theta}{1 + \rho} y_t^*;$$

with repayments

$$v_{t+1}^* = (1 + \rho)d_t^*.$$

In a world where debt repayments can be enforced at no further cost, borrowers do not default on their debts; and competitive capital markets achieve an efficient path. Borrowers compete for funds to invest. The demand for borrowing is finite only if the interest rate equals

the return on capital every period. At these prices, individuals with access to technology borrow; the rest lend their savings.

Property 4 (Competitive equilibrium). *Competitive equilibrium attains an efficient allocation, which corresponds to $\gamma^c(h) = e_{h0}/e_0$, i.e.*

$$c_{ht}^c = \frac{e_{h0}}{e_0} c_t^*; \quad d_t^c = \frac{e_L}{e_0} k_t^*$$

where $e_L = \int_0^\pi e_{h0} dh$. Any efficient path can be achieved as the competitive equilibrium of an economy following some redistribution of endowments.

The volume of trade in capital markets is a positive, and constant fraction of national income. It grows at the rate θ , as do consumption and total investment. This is a natural outcome if debt repayment can be enforced at no further cost to lenders. Our interest lies in environments where this is not true, which we turn to next.

4. Default and penalties

The possibility and implications of default are easy to describe. Imagine that an individual without access to the productive technology lends an amount d to individual who has access. The latter invests this amount and, next period, is in possession of investment income $d(1 + \rho)$ and owes the promised amount v to the lender. It is possible to default, and simply not pay v . If there are no methods of enforcing repayment, or penalties to deter default, a rational borrower will, indeed, renege whenever $v > 0$: in anticipation of this, there can be no lending.

We consider two ways to sustain debt repayment. The first, called *capture*, describes direct, but partial, enforcement. A creditor can capture a fraction of the investment income from her funds. Let $0 < \lambda \leq 1$ be this capture rate. A lender who lends d_t at t can capture $\lambda(1 + \rho)d_t$ at $t + 1$. Accordingly, a borrower in default can only abscond with $(1 - \lambda)(1 + \rho)d_t$. Importantly, we assume that a creditor cannot capture the income from private investments by the borrower, or, indeed, that from other creditors' funds. The facility of capture is only available for one period, and lapses thereafter. This is natural in a world where production takes one-period, and capital depreciates fully. We consider an additional penalty imposed on debtors in default, which is *exclusion*. A borrower who defaults on his loans cannot borrow again.

In the following, we characterize the extent to which default-free trade can be enforced by a combination of these methods. The facility of capture can sustain some amount of borrowing and lending whenever $\lambda > 0$. Borrowers do not gain from default if repayments are $v_{t+1} \leq d_t(1 + \rho)\lambda$. We examine the incremental value of the exclusion penalty. The argument is constructive, and in several steps. For exclusion to be a viable penalty, some individuals must have the right to borrow in the first place. We denote this by $b_{ht} \in \{0, 1\}$, where $b_{ht} = 1$ indicates that individual h has the right to borrow at time t . Borrowing rights are allocated to some individuals with access: $b_{h0} = 1 \Rightarrow h \geq \pi$. A borrower can choose to default at any time, t : this is a choice of action $\delta_{ht} \in \{0, 1\}$, where

1 corresponds to default. The penalty of exclusion is the rule that borrowing rights cease with default:

$$b_{ht} = b_{h,t-1}(1 - \delta_{ht}).$$

Lenders, similarly, can exercise their right of capture, indicated by $\sigma_{ht} \in \{0, 1\}$. With, $\sigma_t = 1$ indicates capture at t .

A debt plan, which specifies levels of debt and repayment to borrowers every period, is said to be sustained if borrowers do not default: $\delta_{ht}b_{ht} = 0$ at all $t \geq 1$; and lenders do not capture the investment income: $\sigma_{ht}(1 - b_{ht}) = 0$ for each $t \geq 1$. In the rest of this Section, we characterize the property of debt contracts and plans that can be sustained by the twin penalties of capture and exclusion.

4.1. Lending and capture

Individuals without access are potential lenders. Let $i \in [0, \pi)$ denote such a person, who lends her entire savings.¹ In order to choose her consumption-savings path, she needs a list of repayments associated with each possible level of savings. This is the *debt contract* at t , $v_{i,t+1}(s)$, which promises repayment levels $v_{i,t+1}(\cdot)$ at $t + 1$ for each non-negative level of savings at t . We note that these are one-period debt contracts; and that they may be personalized, with $v_i(s) \neq v_{i'}(s)$.

At each t , lender i chooses σ_{it} —whether or not to capture her funds—as well as her consumption and savings levels, c_{it}, s_{it} . Her incomes are

$$x_{it} = (1 - \sigma_{it})v_{i,t}(s_{i,t-1}) + \sigma_{it} \lambda(1 + \rho)s_{i,t-1}$$

at each $t \geq 1$. Budget constraints are

$$s_{it} + c_{it} \leq x_{it}, \quad t \geq 1;$$

with $c_{it}, s_{it} \geq 0$; the last restriction reflects the fact that she cannot borrow, and finances consumption and further savings from current income.

Lemma 1. *Let $0 \leq i < \pi$, and $b_{i0} = 0$. Lender i does not capture her funds at any time, i.e. $\sigma_{it} = 0$ for each $t \geq 1$ if, and only if*

$$v_{it}(s_{i,t-1}) \geq \lambda(1 + \rho)s_{i,t-1} \quad (\mathbf{I}_i)$$

at each $t \geq 1$.

Lemma 1 is self-evident: capture rights are exercised only if they increase income, in the period where repayment is due. The only substantive issue is that **(I_i)** restricts the repayment schedules $v_{i,t+1}(\cdot)$ at the points s_{it} , rather than globally.

¹ Individuals with access to technology can invest rather than lend, if denied borrowing rights; they will do so because investment yields at least as much as debt.

4.2. Borrowing and default

Some individuals j , with access to the technology have the right to borrow: $b_{j0} = 1 \Rightarrow \pi \leq j \leq 1$. Any such borrower foresees a *debt plan*, specifying amounts to be borrowed, as d_{jt} , and repayments, as $v_{j,t+1}$, for each $t \geq 0$. Let $Z_{j0} = (D_{j0}, V_{j1}) = \{\dots z_t = (d_{jt}, v_{j,t+1}), \dots\}$ be the debt plan available to j . At each t , he chooses $\delta_{jt} \in \{0, 1\}$ —whether or not to default, as well as his consumption and investment levels c_{jt}, k_{jt} . Borrowing rights cease with default:

$$b_{j0} = 1; \quad b_{jt} = b_{j,t-1}(1 - \delta_{jt}), \quad \text{for } t \geq 1.$$

Actual borrowing is

$$\tilde{d}_{jt} = d_{jt}b_{jt}.$$

The debt plan Z_{j0} generates earnings x_{jt} :

$$x_{jt} = (1 - \delta_{jt})[(1 + \rho)\tilde{d}_{j,t-1} - v_{jt}] + \delta_{jt}(1 - \lambda)(1 + \rho)\tilde{d}_{j,t-1}; \quad t \geq 1.$$

Borrower’s budget constraints are

$$c_{jt} + I_{jt} = (1 + \rho)I_{j,t-1} + x_{j,t}, \quad \text{for } t \geq 1;$$

with $c_{jt}, I_{jt} \geq 0$. We emphasize that I_{jt} refers to the borrower’s private investment, which cannot be confiscated in the event of default. Thus, incentives to default or repay at t depend entirely on the income stream X_{jt} . We have written the budget constraints to maintain this distinction between incomes that are susceptible to confiscation. Total investment in j ’s technology is $k_{jt} = \tilde{d}_{jt} + I_{jt}$.

Lemma 2. *Let $\pi \leq j \leq 1$, and $b_{j0} = 1$. Borrower j does not default at any time, i.e. $\delta_{jt} = 0$ at each $t \geq 1$, if, and only if Z_{j0} satisfies*

$$(1 - \lambda)(1 + \rho) d_{j,t-1} \leq \sum_{i=0}^{\infty} \frac{d_{j,t-1+i}(1 + \rho) - v_{j,t}}{(1 + \rho)^i} \equiv X_{jt}^+ \quad (\mathbf{I}_j)$$

at each $t \geq 1$.

Proof. We note, first, that borrower j has access to the productive technology; and compares any pair of income streams by their present values. Suppose $b_{jt} = 1$. Let $X_t^+(\delta_{jt})$ be the value of income streams resulting from actions $\delta_{jt} \in \{0, 1\}$. From the definitions,

$$X_t^+(1) = d_{j,t-1}(1 + \rho)(1 - \lambda);$$

and

$$X_t^+(0) = d_{j,t-1}(1 + \rho) - v_{j,t} + \frac{\max[X_{t+1}^+(0), X_{t+1}^+(1)]}{1 + \rho}.$$

□

Borrower j chooses to repay if $X_t^+(0) \geq X_t^+(1)$. If $X_{t+i}^+(0) \geq X_{t+i}^+(1)$ for all $i \geq 0$, we have condition (\mathbf{I}_j) , and that

$$X_{jt}^+ = X_{jt}^+(0) = (1 + \rho)D_{j,t-1}^+ - V_{j,t}^+$$

as stated.

A borrower who never defaults earns $x_{jt} = (1 + \rho)d_{jt-1} - v_{jt}$ each period. Borrowers choose to repay debts in order to maintain the right to borrow. This right is valuable because it generates positive incomes: X_{jt}^+ is the value of the debt plan Z_{jt} . We note that (\mathbf{I}_j) implies $X_{jt}^+ \geq (1 - \lambda)(1 + \rho)d_{j,t-1}$, which is strictly positive whenever $d_{j,t-1} > 0$ and $\lambda < 1$.

Competitive capital markets generate debt plans with $v_{j,t+1}^c = d_{jt}(1 + \rho)$: this plan cannot sustain default-free debt with partial enforcement.

Corollary 1. *A competitive debt plan satisfies*

$$v_{t+1}^c(d) = (1 + \rho)d.$$

This plan is consistent with default-free debt, $D_0^c > 0$ if, and only if, $\lambda = 1$.

A competitive debt plan generates zero incomes for borrowers: $x_{j,t+1} = 0 \Rightarrow X_{jt}^+ = 0$ for all j . Thus, $\lambda < 1 \Rightarrow d_{jt} = 0$ for each j and t .

Bulow and Rogoff (1989) emphasize this in the context of sovereign debt. If the borrower is a sovereign contry, it is able to retain all investment income in default, so that $\lambda = 0$. Corollary 1 establishes that their main proposition—that competitive trade in capital markets cannot be sustained without default—is true whenever $\lambda < 1$.

5. Sustainable paths

5.1. Debt

Lenders face one-period debt contracts and choose their savings. Borrowers face individual debt plans, which specify borrowing and repayments. The two sides must balance in the aggregate. A debt allocation \mathcal{B}_0 is a list of savings and debt contracts for lenders, and borrowing rights and debt plans for borrowers:

$$\mathcal{B}_0 = [\{s_{i0}, V_{i1}(\cdot); 0 \leq i < \pi\}, \{b_{j0}, Z_{j0} = (D_{j0}, V_{j1}); \pi \leq j \leq 1\}]$$

such that

$$s_{i0} \geq 0; b_{j0} \in \{0, 1\}, D_{j0} \geq 0, \int_{\pi}^1 b_{j0} dh > 0.$$

This allocation generates an *aggregate debt plan* $Z_0 = (D_0, V_1) = \{(\dots, z_t = (d_t, v_{t+1}), \dots)\}$. The debt allocation \mathcal{B}_0 is *balanced* if Z_0 satisfies

$$\int_0^{\pi} s_{it} dh = d_t = \int_{\pi}^1 d_{jt} b_{jt} dh;$$

$$\int_0^{\pi} v_{i,t+1}(s_{it}) dh = v_{t+1} = \int_{\pi}^1 v_{i,t+1} b_{it} dh;$$

for each $t \geq 0$.

A debt allocation can be sustained if lenders do not capture and borrowers do not default at any point. We say that a debt allocation \mathcal{B}_0 is λ -sustainable whenever $\int_0^\pi \sigma_{it} dh = 0$ and $\int_\pi^1 \delta_{jt} b_{j0} dh = 0$ at each $t \geq 1$.

We focus on *proportional* debt-allocations, where every designated lender gets a stationary proportion of the aggregate debt plan:

$$D_{j0} = w_j D_0; \quad V_{j1} = w_j V_1; \quad 1 \geq w_j b_{j0} > 0.$$

Balance implies $\int_\pi^1 w_j b_{j0} dj = 1$. **Theorem 1** characterizes sustainable debt-allocations, which restricts the associated aggregate debt plan Z_0 .

Theorem 1 (Sustainable debt). *Let \mathcal{B}_0 be a balanced debt allocation and $Z_0 = (D_0, V_1)$ the associated aggregate debt plan. \mathcal{B}_0 is λ -sustainable only if*

$$v_{t+1} \geq \lambda(1 + \rho) d_t \geq V_{t+1}^+ - D_{t+1}^+ \quad (\mathbf{SD})$$

at each $t \geq 0$. A balanced proportional debt allocation \mathcal{B}_0 is λ -sustainable if it satisfies **(SD)**.

Proof. Let \mathcal{B}_0 be a balanced debt allocation. From **Lemma 1**,

$$\sigma_{i,t+1} = 0 \Leftrightarrow v_{i,t+1}(s_{it}) \geq \lambda(1 + \rho)s_{it}$$

for each $t \geq 0$. By aggregation,

$$\int_0^\pi \sigma_{it} dh = 0 \Leftrightarrow v_{t+1} \geq \lambda(1 + \rho)d_t.$$

From **Lemma 2**,

$$\delta_{j,t+1} = 0 \Leftrightarrow (1 - \lambda)(1 + \rho) d_{jt} \leq (1 + \rho)D_{jt}^+ - V_{j,t+1}^+$$

for $t \geq 0$. By aggregation,

$$\int_\pi^1 \delta_{jt} b_{j0} dh = 1 \Rightarrow (1 - \lambda)(1 + \rho)d_t \leq (1 + \rho)D_t^+ - V_{t+1}^+$$

for each $t \geq 1$. Further, $(1 + \rho)D_t^+ = (1 + \rho) d_t + D_{t+1}^+$ by definition. Substitution yields **(SD)**.

Suppose Z_0 satisfies **(SD)**, and individual debt plans are proportional: $Z_{j0} = w_j Z_0$. For each $w_j > 0$, Z_{j0} satisfies **(SD)** if, and only if Z_0 does. \square

The restriction to proportional debt allocations is important for the following reason. It is possible to imagine situations where Z_{jt} is an increasing proportion of Z_t : a borrower is deterred from default by the prospect of an increasing market. This cannot be true for all borrowers in the economy. Because of aggregate feasibility, other borrowers must face the prospect of losing their market share, and may, then, have greater incentives to default.

This offers a slightly different interpretation of **Theorem 1**. Suppose the aggregate debt plan satisfies condition **(SD)**. Recall that the condition is necessary for sustainability, and

cannot be weakened. The theorem displays that it is always possible to find a division of the debt market among potential borrowers that sustains the plan. This division involves more than an allocation of borrowing rights, because each designated borrower obtains the right to service a fixed fraction of the population.

5.2. Allocations

An allocation C_0 is a complete set of consumption plans. From [Property 1](#), it is feasible whenever the aggregate consumption plan satisfies $C_0^+ \leq e_0$. Not every such path can be sustained by default-free debt. A feasible allocation is λ -sustainable if it can be financed by a λ -sustainable debt plan Z_0 .

We recall that a lender’s consumption c_{jt} is financed by repayments net of debt. Restrictions on Z_0 imply constraints on their consumption plans. The restriction **(SD)** on aggregate debt constrains the aggregate consumption levels of lenders as follows. Let $c_{Lt} = \int_0^\pi c_{ht} dh$ be the aggregate consumption level of lenders. We have

$$c_{Lt} \leq v_t - d_t, \quad \text{for } t \geq 1$$

Feasible paths satisfy

$$c_t \leq k_{t-1}(1 + \rho) - k_t = d_{t-1}(1 + \rho) - d_t + \int_\pi^1 [I_{j,t-1}(1 + \rho) - I_{jt}]dh;$$

for $t \geq 1$, and

$$c_0 + d_0 + \int_\pi^1 I_{j0} dj \leq e_0.$$

[Theorem 2](#) characterizes λ -sustainable allocations.

Theorem 2 (Sustainable consumption). *A feasible allocation, C_0 , is λ -sustainable if, and only if $C_{L0} = \{ \dots, c_{Lt} = \int_0^\pi c_{it} di, \dots \}$ satisfies*

$$\text{(SC)} \quad C_{L0}^+ - (1 - \lambda)c_{L0} = \lambda c_{L0} + \sum_{t=1}^\infty \frac{c_{Lt}}{(1 + \rho)^t} \leq \lambda e_0.$$

Proof. We note that $d_0 \leq e_0 - c_{L0}$ because $c_{h0}, k_{h0} \geq 0$.

Suppose that **(SC)** fails, and $C_{L1}^+ > \lambda(1 + \rho)(e_0 - c_{L0}) \geq \lambda(1 + \rho)d_0$. Further, $c_{Lt} \leq v_t - d_t$ for $t \geq 1$. Thus, $V_1^+ - D_1^+ \geq C_{L1}^+ > \lambda(1 + \rho)d_0$, and **(SD)** fails at $t = 0$ for any debt plan that finances c_{Lt} . This establishes the necessity of **(SC)**.

Suppose, next, that C_{L0} satisfies **(SC)**. We construct an aggregate debt-plan Z_0 that finances this path, and satisfies **(SD)**. Define

$$\hat{d}_0 = \frac{C_{L1}^+}{\lambda(1 + \rho)}.$$

Note that **(SC)** ensures $c_{L0} + \hat{d}_0 \leq e_0$. Define

$$\hat{v}_t = (1 + \rho)d_{t-1} - \frac{1 - \lambda}{\lambda}c_{Lt};$$

$$\hat{d}_t = \hat{v}_t - c_{Lt} = (1 + \rho) d_{t-1} - \frac{c_{Lt}}{\lambda}.$$

By recursion,

$$\hat{d}_t = \frac{(1 + \rho)^t}{\lambda} \left[\lambda \hat{d}_0 - \sum_{i=1}^t \frac{c_{Li}}{(1 + \rho)^i} \right] = \frac{C_{L,t+1}^+}{\lambda(1 + \rho)} \geq 0.$$

Further,

$$\hat{v}_t - \lambda(1 + \rho) \hat{d}_{t-1} = (1 - \lambda) \hat{d}_t \geq 0.$$

Finally,

$$\lambda(1 + \rho) \hat{d}_t - (\hat{V}_{t+1}^+ - \hat{D}_{t+1}^+) = (1 + \rho)^t (\lambda(1 + \rho) \hat{d}_0 - C_{L,1}^+) = 0.$$

This verifies that \hat{Z}_0 satisfies **(SD)**, and proves the assertion. □

The set of λ -sustainable allocations exhibits some properties that are useful in further analysis.

Corollary 2 (Convexity). *The set of λ -sustainable allocation is convex.*

(SC) is a linear restriction on C_{L0} , implying convexity.

Corollary 3 (Monotonicity). *The set of sustainable allocations is increasing in λ .*

Let $0 < \lambda < \lambda' \leq 1$. An allocation is λ -sustainable whenever

$$C_{L1}^+ \leq \lambda(1 + \rho)(e_0 - c_{L0}).$$

We note that $e_0 - c_{L0} \geq 0$ for feasible allocations, implying $\lambda'(1 + \rho)(e_0 - c_{L0}) > \lambda(1 + \rho)(e_0 - c_{L0}) \geq C_{L1}^+$; it is necessarily λ' -sustainable.

Corollary 4 (Positivity). *Sustainable consumption paths for lenders are positive for $t \geq 1$ only if $\lambda > 0$.*

If $\lambda = 0$, the condition **(SC)** implies $C_{L1}^+ \leq 0$, which implies $c_{Lt} = 0$ for all $t \geq 1$.

6. Constrained efficiency

Sustainable allocations must satisfy a single constraint, **(SC)**, which restricts the aggregate consumption path of lenders. The set of such allocations is convex. An allocation C_0 is constrained efficient if it solves the program

$$\max \int V(C_{h0}) \omega(h) dF(h) \quad (\mathbf{P}_c)$$

subject to feasibility and λ -sustainability constraints **(F)** and **(SC)**. Solutions are second-best paths, which may differ from first-best ones. We show, in **Theorem 3**, that a subset of efficient allocations satisfy the constraint **(SC)**, and can thus be sustained by default-free debt.

Along any efficient path,

$$c_{L0}^* = \gamma_L \frac{\rho - \theta}{1 + \rho} e_0;$$

$$c_{Lt}^* = c_{L0}^* (1 + \theta)^t;$$

with

$$\gamma_L = \frac{\int_0^\pi \omega(h)^{1/\sigma} dh}{\int_0^1 \omega(h)^{1/\sigma} dh}.$$

Theorem 3 (Some efficient paths are sustainable). *An efficient allocation is λ -sustainable if, and only if*

$$c_{L0}^* \leq e_0 \frac{\lambda(\rho - \theta)}{\lambda(1 + \rho) + (1 - \lambda)(1 + \theta)};$$

this is true whenever

$$\gamma_L \leq g(\lambda) \equiv \frac{\lambda(1 + \rho)}{(1 + \rho)\lambda + (1 - \lambda)(1 + \theta)}.$$

Proof. We need to verify (SC), which is true if, and only if

$$C_{L0}^* - (1 - \lambda)c_{L0}^* \leq \lambda e_0.$$

Along an efficient path, $c_{Lt}^* = (1 + \theta)^t c_{L0}^* \Rightarrow C_{L0}^* = (1 + \rho)/(\rho - \theta)c_{L0}^* = \gamma_L e_0$. Substitution completes the proof, noting that $\rho > \theta$ from Assumption 3. \square

From Corollary 4, a consumption plan for lenders is positive from $t = 1$ on only if $\lambda > 0$. Theorem 3 establishes that this condition guarantees that some efficient paths can be sustained. We know that the set of sustainable allocations are monotonic in λ ; this property is inherited by their efficient subsets, as $g(\lambda)$ increases with λ . It coincides with the set of fully efficient allocations when $\lambda = 1$, because $g(1) = 1$.

Corollary 5. *The allocation C_c^0 , generated at a competitive equilibrium, is λ -sustainable if, and only if*

$$e_{L0} \leq g(\lambda)e_0.$$

The allocation achieved by a competitive equilibrium corresponds to $\gamma_L^c = e_L/e_0$. This can be sustained if e_L is small enough, or if λ is large enough.

Corollary 1 showed that the competitive debt contract cannot be sustained if $\lambda < 1$; while Corollary 5 shows that the allocation is sustainable for some $\lambda < 1$. This would appear to be paradoxical. The solution lies in the nature of the debt contract supporting an allocation in the presence of default possibilities. The competitive debt plan $v_t^*(d_t) = (1 + \rho)d_t$ is not λ -sustainable; however, a different debt contract, and associated plan, can support the same allocation. We now consider the nature of debt-contracts that support default-free trades, with particular attention to those that support fully-efficient allocations.

7. Debt contracts and decentralization

We know that competitive equilibrium achieve efficient paths whenever $\lambda < 1$. The associated debt contracts are of the form $v_{t+1}^c = (1 + \rho)d_t$. Interest rates index the price of bonds, and competition in capital markets drives the rate of interest to equal the highest rate of return on capital. In this section, we ask whether it is possible to decentralise constrained efficient allocations with $\lambda < 1$. From [Theorem 3](#), we know that some efficient can be sustained by default-free trade as long as $\lambda > 0$, and also, that the associated debt contracts must be different from v^c .

We evaluate the possibility of decentralising such allocations in a private economy. Specifically, we consider a candidate allocation $\hat{C} = [\hat{C}_{h0}; 0 \leq h \leq 1]$ that is efficient and λ -sustainable, and construct a family of debt contracts and reallocation of initial endowments and borrowing rights, that yields an equilibrium coinciding with the chosen allocation.

In doing this, we need to pay attention to the nature of debt contracts: these must generate sufficient rents in order to deter default, as well as achieve efficient consumption smoothing by borrowers and lenders. Borrowing generally generates rents, and borrowing rights need to be restricted. They are given to the individuals with access to productive technology, and may be further restricted as part of the penalty of default. The description of the economy must account for these differential rights to borrow and lend, as well as endowment redistributions familiar from classical welfare economics.

The private economy, $\mathcal{E} = (\mathcal{A}, \mathcal{V}, \mathcal{W})$ is specified by

1. A distribution of initial endowments

$$\mathcal{A} = \left[a_{h0} \geq 0; 0 \leq h \leq 1; \int_0^1 a_{h0} dh \leq e_0 \right];$$

2. A set of debt contracts

$$\mathcal{V} = [\{v_{h,t+1}(s); s \geq 0\}_{t=0}^\infty; 0 \leq h \leq 1];$$

3. An allocation of borrowing rights

$$\mathcal{W} = \left[w_{h0} \geq 0; \int_0^1 w_j dh = 1; 0 \leq h \leq 1 \right],$$

with the exclusion penalty $w_{h,t+1} = (1 - \delta_{ht})w_{h,t}$, where $\delta_{ht} = 1$ indicates default by h at t .

Individuals choose consumption and savings every period, borrowers with $w_j > 0$, also choose their default decisions $\delta_{jt} \in \{0, 1\}$. An allocation of \mathcal{C}_0 is an equilibrium of this economy if, and only if, there is no default, i.e. $\delta_{jt} = 0$ at each $t \geq 1$ whenever $w_{j0} > 0$; and C_{h0} maximizes utility subject to budget and borrowing constraints for each $h \in [0, 1]$. Debt contracts $\hat{\mathcal{V}}$ are said to support an allocation $\hat{\mathcal{C}}_0$ if it is an equilibrium of some private economy $\mathcal{E} = (\mathcal{A}, \hat{\mathcal{V}}, \mathcal{W})$.

7.1. Debt contracts

An economy \mathcal{E} may have many equilibria; similarly, a candidate allocation may be an equilibrium in more than one economy. Here, we provide a constructive argument to find

an economy $\hat{\mathcal{E}}$, with debt contracts $\hat{\mathcal{V}}$, that posses an efficient, λ -sustainable equilibrium $\hat{\mathcal{C}}$. Efficiency implies $\hat{c}_{ht} = (1 + \theta)^t \hat{c}_{h0}$, and λ -sustainability requires $\hat{c}_{L0} \leq e_0(\lambda(1 + \rho)/\lambda(1 + \rho) + (1 - \lambda)(1 + \theta))$.

Supporting debt contracts $\hat{\mathcal{V}}$ must perform several functions. Lenders choose their consumption path by maximizing utility, subject to budget constraints defined directly by debt contracts. These consumption paths must coincide with $\hat{\mathcal{C}}_i$. Lenders' choices of savings generate an aggregate debt plan Z_0 , that needs to be default-free. The incomes or fees, generated by the debt contracts must be large enough to induce repayment. Finally, repayment on debt has to finance current and future consumption, as well as promised fees.

Consider an allocation $\hat{\mathcal{C}}$ with $\hat{\mathcal{C}}_i > 0$. For a lender i , facing $v_{it}(\cdot)$, a consumption path $\hat{\mathcal{C}}_i > 0$ is individually rational only if it satisfies the first order condition for utility maximization

$$\frac{U'(\hat{c}_{i,t})}{U'(\hat{c}_{i,t+1})} = \beta \frac{\partial v_{i,t+1}}{\partial d} \Big|_{d=d_{it}}$$

at each $t \geq 0$. Along an efficient path, at each t ,

$$\frac{U'(\hat{c}_{i,t})}{U'(\hat{c}_{i,t+1})} = \beta(1 + \rho).$$

We consider a class of debt contracts that meet this requirement by construction. Specifically, these are debt contracts with lump-sum fees. An individual i is required to pay a fee, x_{it} , if she wishes to lend a positive amount at time t , irrespective of the amount. The associated debt contracts are

$$v_{i,t}(d_{i,t-1}) = \begin{cases} (1 + \rho)d_{i,t-1}, & \text{if } d_{i,t} = 0 \\ (1 + \rho)d_{i,t-1} - x_{it}, & \text{if } d_{i,t} > 0 \end{cases}$$

at each $t \geq 1$. Fees paid by lenders, and collected by designated borrowers. The typical debt contract is in two parts, specifying the schedule of fees $\mathcal{X} = [X_{h0}; h \in [0, 1]]$, and the marginal rate of return on debt, ρ . We refer to these contracts as $\mathcal{V}(\mathcal{X} \leftrightarrow \rho)$. Debt contracts are personalised, in that fees can vary across individuals or over time. They are independent of the amounts actually lent; we note that the resulting debt contracts depends on d_{it-1} , d_{it} .

Lemma 3 establishes the level of fees that is sufficient to deter default by borrowers.

Lemma 3 (Participation fees). *A set of debt contracts with fees, $\mathcal{V}(\mathcal{X}, \rho)$ can support an efficient allocation $\hat{\mathcal{C}}$ only if $X_{L0} = \{ \dots, \int_0^\pi x_{it} dF(i), \dots \}$ satisfies*

$$X_{L0}^+ \geq \frac{1 - \lambda}{\lambda} \hat{\mathcal{C}}_{L0}^+ = \frac{1 - \lambda}{\lambda} \frac{1 + \rho}{\rho - \theta} \hat{c}_{L0}$$

at each $t \geq 1$, where $\hat{c}_{L0} = \int_0^\pi \hat{\mathcal{C}}_{it} dh$

Proof. The contracts $\hat{\mathcal{V}}$ imply the aggregate debt plan $\hat{Z}_0 = (\hat{V}_1, \hat{D}_0)$ as follows. Let $a_L = \int_0^\pi a_{h0} dh$, and

$$\begin{aligned} \hat{d}_0 &= a_L - x_{L0} - \hat{c}_{L0}; & \hat{d}_t &= (1 + \rho)\hat{d}_{t-1} - x_{Lt} - \hat{c}_{Lt}; \\ \hat{v}_t &= (1 + \rho)\hat{d}_{t-1} - x_{Lt} = \hat{d}_t + \hat{c}_{Lt}. \end{aligned}$$

We obtain

$$\hat{d}_t = \frac{\hat{C}_{L,t+1}^+ + X_{L,t+1}^+}{1 + \rho}$$

for each $t \geq 0$. Further, $V_t^+ - D_t^+ = \hat{C}_{L,t}^+$. Default-free trades require **(SD)** at each $t \geq 1$, implying

$$\hat{C}_{L,t}^+ + X_{L,t}^+ - x_t \geq \lambda(\hat{C}_{L,t}^+ + \hat{X}_{L,t}^+) \geq \hat{C}_{L,t}^+.$$

The second inequality implies $X_{L,t}^+ \geq ((1 - \lambda)/\lambda)\hat{C}_{L,t}^+$. □

The minimum fee that deters default can be quite large if λ is near zero. The natural question, then, is whether potential lenders will agree to pay this fee to participate in capital markets. **Lemma 4** derives a necessary condition for the participation.

Lemma 4 (Participation). *Let \hat{C} be an efficient and λ -sustainable allocation, with $\hat{C}_{it} > 0$ for each $i \in [0, \pi)$. This allocation can be supported by debt contracts with fees only if $D_{i0} > 0$ for each $i \in [0, \pi)$, implying*

$$\lambda^{1-\sigma} \geq \left[\frac{\rho - \theta}{1 + \rho} \right]^\sigma \quad (\mathbf{Q}).$$

Proof. We note, first, that $\lambda \in (0, 1]$ and $(\rho - \theta)/(1 + \rho) < 1$. Thus, condition **(Q)** holds trivially whenever $\sigma \geq 1$.

The contracts \hat{V} support \hat{C} only if \hat{C}_i is the optimal consumption path for individual i . From **Lemma 3**, $d_{it} = (X_{i,t+1}^+ + C_{i,t+1}^+)/ (1 + \rho) \geq (C_{i,t+1}^+)/ (1 + \rho)$. A strictly positive consumption path implies $d_{it+1} > 0$ at each t . We need to establish whether positive lending is individually rational, given any path of fees satisfying **(SD)**.

For any individual i , the choice $d_{it} = 0$ at time t implies $c_{i,t+n} = 0$ for $n \geq 1$. The best consumption path generated by zero savings is

$$\tilde{C}_{it} = \{(1 + \rho)d_{i,t-1}, 0, \dots, 0 \dots\}$$

The utility of this path is

$$V_t(\tilde{C}_{it}) = \begin{cases} U((1 + \rho)d_{i,t-1}), & \text{if } 0 \leq \sigma < 1 \\ -\infty, & \text{if } \sigma \geq 1 \end{cases}$$

The utility of continuing with consumption plan \hat{C}_{it} is

$$V_t(\hat{C}_{it}) = U(\hat{C}_{it}) \left[\frac{1 + \rho}{\rho - \theta} \right].$$

Lenders prefer $d_{it} > 0$ if $V_t(\tilde{C}_{it}) \leq V_t(\hat{C}_{it})$. This is necessarily true whenever $\sigma \geq 1$. For $\sigma < 1$, we obtain

$$V_t(\tilde{C}_{it}) \leq V_t(\hat{C}_{it}) \Leftrightarrow (1 + \rho)d_{i,t-1} \leq \hat{C}_{it} \left[\frac{1 + \rho}{\rho - \theta} \right]^{1/1-\sigma}$$

for each $i \in [0, \pi]$. By aggregation,

$$(1 + \rho)d_{t-1} \leq \hat{C}_{Lt} \left[\frac{1 + \rho}{\rho - \theta} \right]^{1/1-\sigma} \tag{Q1}$$

From [Theorem 2](#), \hat{C}_{L0} is λ -sustainable only if

$$\hat{C}_{Lt}^+ \leq \lambda(1 + \rho)d_{t-1}.$$

\hat{C} is efficient, implying $\hat{C}_{Lt}^+ = \hat{C}_{Lt}[(1 + \rho)/(\rho - \theta)]$, and

$$\hat{C}_{Lt} \leq \lambda(\rho - \theta)d_{t-1} \tag{Q2}.$$

The inequalities [\(Q1\)](#) and [\(Q2\)](#) imply [\(Q\)](#). To complete the proof, observe that $\sigma \geq 1$ implies [\(Q\)](#) necessarily holds and also that $d_t > 0$ at each t . □

Condition [\(Q\)](#) is necessary if two-part debt contracts are to support an efficient allocation. As we note in the proof, this condition necessarily holds whenever $\sigma \geq 1$. The condition fails if $\sigma < 1$, and if λ is sufficiently small. Participation fees need to be large if λ is close to zero. Lenders are unwilling to pay these fees if current consumption is close substitute for future consumption which is true if $\sigma < 1$. We note, in passing, that condition [\(Q\)](#) necessarily fails for each $\lambda < 1$ whenever $\sigma = 0$, implying perfect substitution between current and future consumption.

7.2. Decentralisation

We introduced a class of debt contracts with fees, as $\mathcal{V}(\mathcal{X}, \rho)$. Here, we demonstrate that, if [\(Q\)](#) holds, any allocation \hat{C} , that is efficient and λ -sustainable can be supported by some debt contract with fees.

Let \hat{C} be an efficient and λ -sustainable allocation. Define a collection of entry fees as

$$\hat{x}_{ht} = \frac{(1 - \lambda)}{\lambda} \hat{c}_{ht}; \quad 0 \leq h \leq 1, \quad t \geq 0.$$

The proposed debt contract is $\hat{\mathcal{V}} = \mathcal{V}(\hat{\mathcal{X}}, \rho)$, i.e.

$$\hat{v}_{h,t} \begin{cases} (1 + \rho)d_{h,t-1}, & \text{if } d_{h,t} = 0 \\ (1 + \rho)d_{h,t-1} - \hat{x}_{ht}, & \text{if } d_{h,t} > 0, \end{cases}$$

at each t . In **Theorem 4**, we show that $\hat{\mathcal{V}}$ supports $\hat{\mathcal{C}}$ whenever condition **(Q)** holds: the candidate allocation is an equilibrium of some economy $\hat{\mathcal{E}}$, with a distribution of initial endowments and borrowing rights, and the proposed debt contract.

Theorem 4 (Efficient debt contracts). *Suppose **(Q)** holds. Let $\hat{\mathcal{C}} = [\hat{\mathcal{C}}_{h0}; 0 \leq h \leq 1]$ be an efficient, λ -sustainable allocation. The debt contracts $\hat{\mathcal{V}} = \mathcal{V}(\hat{\mathcal{X}}, \rho)$ support the allocation $\hat{\mathcal{C}}$, with $\hat{X}_{h1} = ((1 - \lambda)/\lambda)\hat{\mathcal{C}}_{h1}$.*

Proof. We have to demonstrate a distribution of endowments, \mathcal{A} , and borrowing rights, \mathcal{W} , such that $\hat{\mathcal{C}}$ is an equilibrium of the economy $\mathcal{E} = (\mathcal{A}, \hat{\mathcal{V}}, \mathcal{W})$. Recall that the candidate allocation satisfies $\hat{c}_{ht} = (1 + \theta)^t \hat{c}_{h0}$ and $c_{Lt} \leq g(\lambda)\hat{c}_t$. Let

$$\hat{X}_{h0} = \left\{ 0, \frac{1 - \lambda}{\lambda} \hat{\mathcal{C}}_{h1} \right\};$$

and $x_{Lt} = \int_0^\pi x_{ht} dh$. We note that $\hat{X}_{Lt}^+ = ((1 - \lambda)/\lambda)[(1 + \theta)/(\rho - \theta)]\hat{\mathcal{C}}_{L0}$. □

1. *Initial allocation:* Define an allocation of endowments, $\hat{\mathcal{A}}$, and borrowing rights, $\hat{\mathcal{W}}$ as:

$$0 \leq i < \pi \Rightarrow \hat{w}_i = 0; \quad \hat{a}_i = \hat{\mathcal{C}}_{i0}^+ + \hat{X}_{i0}^+; \quad \pi \leq j \leq 1 \Rightarrow \hat{w}_j = \frac{\hat{c}_{h0}}{\int_\pi^1 \hat{c}_{h0} dh};$$

$$\hat{a}_j = \hat{\mathcal{C}}_{j0}^+ - \hat{w}_j \hat{X}_{L0}^+.$$

This redistribution is feasible, with $\hat{a}_h \geq 0$ and $\int_\pi^1 \hat{a}_{h0} dh$ whenever $c_{L0} \leq g(\lambda)c_0$. We now consider equilibrium in the economy $\hat{\mathcal{E}} = (\hat{\mathcal{A}}, \hat{\mathcal{V}}, \hat{\mathcal{W}})$.

2. *Consumption choices: lenders:* Consider any individual i , with $0 \leq i < \pi$. The candidate consumption path $\hat{\mathcal{C}}_{i0}$ exhausts her budget, and satisfies the first order conditions of utility maximization whenever $C_{i,t+1} > 0 \Rightarrow d_{it} > (\hat{x}_{it}/(1 + \rho)) > 0$. It is the optimal consumption path for each such individual provided $d_{it} > 0$ at each t . Suppose $c_{i,t-n} = \hat{c}_{i,t-n}$ for $1 \leq n \leq t$, and individual i contemplates $d_{it} = 0$: this deviation yields the consumption path $\tilde{\mathcal{C}}_{it} = \{(1 + \rho)d_{i,t-1}, 0, \dots, 0, \dots\}$, and $d_{it} > 0$ whenever $V_t(\hat{\mathcal{C}}_{it}) \geq V_t(\tilde{\mathcal{C}}_{it})$ at each t . Observe that $d_{i,t-1}(1 + \rho) = \hat{\mathcal{C}}_{it}^+ + \hat{X}_{it}^+ = \hat{c}_{it}(1 + \rho)/\lambda(\rho - \theta)$; and that $V_t(\tilde{\mathcal{C}}_{it}) = U(c_{it})(1 + \rho)/(\rho - \theta)$. Further,

$$V_t(\tilde{\mathcal{C}}_{it}) = \begin{cases} U\left(\hat{c}_t \frac{1 + \rho}{\lambda(\rho - \theta)}\right), & \text{if } 0 \leq \sigma < 1 \\ -\infty, & \text{if } \sigma \geq 1. \end{cases}$$

We obtain that **(Q)** $\Rightarrow d_{it} > 0$; both are necessarily true whenever $\sigma \geq 1$. Thus, $\hat{\mathcal{C}}_{it}$ is individually rational for each $i \in [0, \pi)$ whenever **(Q)** is true.

3. *Debt plans:* Consider individual $j \in [\pi, 1]$. He has access to the productive technology which yields $(1 + \rho)s > v_{j,t+1}(s)$, and never chooses to lend, i.e. $d_{jt} = 0$ at each t . Aggregate debt levels are $d_t = \int_0^\pi d_{ht} dh$:

$$d_0 = a_L - \hat{c}_{L0};$$

$$d_t = (1 + \rho)d_{t-1} - \hat{c}_{Lt} - \hat{x}_{Lt} = (1 + \rho)d_{t-1} - \frac{\hat{c}_{Lt}}{\lambda}.$$

This, with $v_{t+1} = (1 + \rho)d_t - ((1 - \lambda)/\lambda)c_{Lt}$, defines the aggregate debt plan, which generates fee incomes X_{L0} for borrowers. The debt-plan satisfies (SD) because $d_t = (c_t^+/\lambda(1 + \rho))$, implying $v_t > \lambda(1 + \rho)d_{t-1} = V_t^+ - D_t^+$, and is therefore default-free.

4. *Consumption choice: borrowers:* Consider individual j , with $\pi \leq j \leq 1$, and borrowing right w_j . This right generates income $w_j X_0$ of value $w_j X_0^+$, and corresponding budget constraint $C_{j0}^+ \leq a_j + w_j X_0^+$. The efficient consumption plan C_{j1} satisfies the first-order conditions of utility maximization; and is budget feasible by construction.

From (2) and (4), the allocation \hat{C}_0 is individually rotational for each $h \in [0, 1]$ whenever (Q) holds. From (3), it is default-free. This completes the argument.

The fee that supports an efficient allocation has several characteristics. First, it is non-anonymous. Two lenders may be charged different fees; and these fees are proportional to their initial savings. Second, the fee income of borrowers is proportional to their share of the debt market. Typically, then, an allocation of borrowing rights gives every borrower a fixed share of the market *and* the right to charge fees in proportion to this share. Finally, the fee increases overtime, being propotional to aggregate income, consumption, and savings.

The fact that the fee is personalised is unpleasant, and it requires loss of anonymity, Corollary 6 establishes that some efficient allocations can be supported by anonymous debt contracts.

Corollary 6 (Efficient anonymous contracts). *Suppose (Q) holds. Let $X_{h0} = X_0\{0, (1 - \lambda)/\pi\lambda\}\hat{C}_{L1}$, define an efficient allocation, \hat{C}_0 , with $\hat{c}_{it} = c_{Lt}/\pi$.*

From Theorem 4, an efficient allocation that satisfies $\hat{C}_{i0} = \hat{C}_{h0}$ for $i, h \in [0, \pi]$ can be supported by an anonymous debt contract $c_{t-1}(s) = (1 + \rho)s - x_{t+1}$ with $x_t = (1 - \lambda/\lambda)\hat{C}_{jt}$. The anonymous contract set out in Corollary 6 is both feasible and efficient; it supports allocations where all lenders have the same consumption paths. We note, from Theorem 4, that this may need a redistribution of initial endowments.

Recall, from Corollary 5, that a competitive equilibrium allocation is λ -sustainable if $(e_L/e_0) < g(\lambda)$. Typically, the supporting debt contracts are personalized, because initial endowments of potential lenders are diverse.

8. Conclusions

- *The debt contract:* In Theorem 4, we show that two-part pricing, consisting of a participation fee and a marginal rate of return on loans, can support all efficient and sustainable allocations. A natural interpretation of this contract is that of “banking fees”. Depositors pay fees to banks, who have the option of declaring bank failure. Banks compete for funds, bidding up the deposits rate to ρ . Importantly, they take fees as given, and depositors are aware that a cut in fees can trigger bank failures. Typically, the fee is non-anonymous, i.e. depositor specific.
- *Anonymity and debt markets:* debt contracts that support constrained optima typically involve lack of anonymity. As this provides natural difficulties of interpretation in decentralization, we evaluate the possibility of anonymous two-part contracts in Corollary 6.

There, as in [Theorem 4](#), lump-sum redistributions are necessary in the initial period. These redistributions are necessary for efficiency, and ensure participation. Otherwise, lenders with low endowments may choose not to save. We do not know, as yet, whether anonymous, non-linear contracts can support efficient allocations without redistributions.

- *Two-part pricing* is routinely used to achieve efficiency in economies with non-convex technologies (e.g. [Brown \(1991\)](#)). It may appear that we use them for similar reasons. This is not true: the set of λ -sustainable allocations is convex. The need for a fixed fee is a direct consequence of the penalty of exclusion. Borrowers need to make rents every period in order not to default. These rents can be generated in more than one way. Fixed fees can generate rents for the borrower without doing violence to the marginal conditions for a lender's optimum. It is likely, then, that a similar construction suffices in other problems involving moral hazard and the threat of termination, including "efficiency wages", or risk-sharing with storage.
- *Optima and equilibria*: We have looked for debt-contracts that support efficient allocations. Obviously, inefficient λ -sustainable allocations may be achieved by other contracts, including more standard debt contracts of the type

$$v_{t+1}(s) = (1 + r_t)s.$$

In related work ([Dutta and Kapur \(1998, 1999\)](#)), we evaluate the restrictions on lending rates r_t that correspond to λ -sustainable debt-plans.

- *The two penalties*: We note that neither penalty is redundant. In the absence of exclusion, effective debt-contracts correspond to capture at each t :

$$v_{t+1}(s) = \lambda(1 + \rho)s;$$

which cannot support first-best paths if $\lambda > 1$. Capture is quite definitely non-redundant, as the set of sustainable allocations increases monotonically with λ ([Corollary 3](#)); so does the set of efficient sustainable allocations ([Theorem 4](#)). From [Corollary 4](#), non-trivial debt and consumption paths are impossible if $\lambda = 0$.

- *Endowments and sustainability*: Our model is special in many ways. While some of these simplify the problem, the endowment condition is critical in deducing **(SC)**. We assume $e_t = 0$ for $t \geq 1$. This condition is useful in focusing attention on the need for lending and borrowing; our characterization of the sustainability condition relies essentially on this property. The general condition, which restricts $C_{L1} - E_{L1}$ similarly, is valid for consumption paths that satisfy $c_{Lt} \geq e_{Lt}$ for $t \geq 1$. This is certainly true if endowments do not increase too fast; modified versions of our results hold if endowments growth is lower than θ , so that perpetual production is necessary to attain efficiency. Further, [Corollary 4](#) may fail with some endowment distributions, as we show in [Dutta and Kapur \(1999\)](#); it is also possible that [Corollary 2](#) fails for some endowment paths.

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References

- Allen, F., 1981. On the prevention of default. *Journal of Finance* 36, 271–276.
- Allen, F., 1985. Repeated principal-agent relationships with lending and borrowing. *Economics Letters* 17, 27–31.
- Atkeson, A., Lucas, R.E., 1992. On efficient distribution with private information. *Review of Economic Studies* 59, 427–453.
- Brown, D.J., 1991. Equilibrium analysis with nonconvex technologies. In: Hildenbrand, W., Sonnenschein, H. (Eds.), *Handbook of Mathematical Economics IV 1963–1992*, North-Holland.
- Bulow, J., Rogoff, K., 1989. Sovereign debt: is to forgive to forget? *American Economic Review* 79, 43–50.
- Coate, S., Ravallion, M., 1993. Reciprocity without commitment: characterization and performance of informal insurance arrangements. *Journal of Development Economics* 40, 1–24.
- Dutta, J., Kapur, S., 1998. Liquidity preference and financial intermediation. *Review of Economic Studies* 65, 551–572.
- Dutta, J., Kapur, S., 1999. Growth, Default, and Intermediation, Discussion Paper 6-99, Birkbeck College.
- Green, E.J., 1987. Lending and smoothing with uninsurable income. In: Prescott, E., Wallace, N. (Eds.), *Contractual Arrangements for Intertemporal Trade*. University of Minnesota Press, Minneapolis, MN.
- Jones, L., Manuelli, R., 1990. A convex model of equilibrium growth: theory and policy implications. *Journal of Political Economy* 98, 1008–1038.
- Kehoe, T., Levine, D., 1993. Debt constrained asset markets. *Review of Economic Studies* 60, 865–888.
- Kimball, M., 1988. Farmers cooperatives as behavior towards risk. *American Economic Review* 78, 224–232.
- Kocherlakota, N., 1996. Implications of efficient risk sharing without commitment. *Review of Economic Studies* 63, 595–609.
- Thomas, J., Worrall, T., 1994. Income fluctuations and asymmetric information: an example of a repeated principal-agent problem. *Journal of Economic Theory* 51, 367–390.