Learning by observation within the firm

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Abstract

This paper explores the implications of learning by observation on the production and wage decisions of a firm. Firms are viewed as collective arrangements that enable workers to share information. This information takes the form of productivity-enhancing innovations and is the product of research by workers. Some workers may choose to free ride on the research of others; and the firm, by its choice of wage profiles, can affect the amount of research done. The production function of a firm is then derived from the technology of learning, as constrained by the incentives of workers. The qualitative properties of the production function – most importantly, the nature of returns to scale and the incentives to limit firm size – are shown to depend on the behaviour and objectives of the firm.

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It is wise to learn from observation, and foolish to learn from experience.

Bengali proverb

1. Introduction

The hypothesis of 'learning by doing' has rather important implications for the economic aspects of technological progress. It is rich enough to generate

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theories of sustained growth in output and incomes, an implication which has been intensively explored in studies of equilibrium aggregate growth (e.g., Romer, 1986; Lucas, 1988). As that literature continues to evolve, it becomes more and more apparent that the predictive properties as well as policy prescriptions of growth models depend on the details of the technological specification: even at the aggregate level, it is not a matter of indifference to know who learns what, from whom, when. This paper is an exploration of one possible method of learning by workers over time and the emphasis is on the fact that new skills and techniques may be learned from others. We examine implications for the behaviour of firms, as well as for the production technology.

This paper starts with a rather roughly translated Bengali proverb, which suggests that ‘doing’ may well be an inefficient method of learning: it is often better to learn from others. If it is possible to observe other people who are doing something reasonably similar, one can hope to imitate their successes, and avoid their failures. Such opportunities for learning are placed here within a model of a firm with several workers. All workers know a less efficient method of production and may spend additional effort to find the more efficient technique (we call this ‘research’). Research is not guaranteed to be successful but there is positive probability that the efficient technique will be found. A worker can also choose to wait for one of his colleagues to be successful, and then imitate her. Observation is assumed costless within the firm, and the imitation flawless. Knowledge is a local public good; so it is no surprise that workers may choose to free ride. Larger firms present greater opportunities for learning, but also increase the incentive of the worker to shirk (and to count upon the success of others to enhance productivity). It becomes clear that research, learning, and discovery take place in a context which involves strategic interaction. This paper explores the effect of such interaction on stochastic productivity growth. To characterize the solution, the model is kept simple. The effect on dynamic growth paths is a subject for further research.

The production function of a firm is derived from the technology of learning, as constrained by the incentives of workers. The qualitative properties of the production function - most importantly, the nature of returns to scale and the incentives to limit firm size - are shown to depend on the behaviour and objectives of the firm. It turns out that the production function either satisfies the efficiency wage hypothesis or has increasing returns to scale. This stems from the fact that wages affect incentives and a wage-setting firm can, by its choice of wage profiles, affect the amount of research done within the organization. Wage payments are made contingent upon the productivity of the worker. From participation constraints knowledgeable workers must be paid their marginal product, but the entry wage can be less than the marginal product from the inefficient technique. In fact, larger firms can pay lower entry wages because they hold out the promise of earlier increases in worker productivity. The efficiency wage region corresponds to situations where the wage profile is relatively flat.
whereas, for a sufficiently steep wage profile, we will have a production function which will be S-shaped, displaying increasing returns up to a point. However, even when production does not display increasing returns, the profits of the wage-setting firm can be strictly increasing in firm size, so that the wage-setting firm will end up being a monopsonist.

We consider alternative models of firm behaviour. A workers' cooperative allows every worker the right to her own produce; a competitive firm is a price-taker on the labour market and chooses employment to maximize profits; a wage-setting firm chooses wages as well as employment to maximize profits and is constrained either by the supply of labour at alternative wage schedules or by overall capacity. As it turns out, the effect of firm behaviour on the production schedule is immaterial either if research is too easy or if it is too difficult. In the intermediate region, different types of firms lead to different choices of wages and of productivity. We show that there is an optimal size distribution of competitive firms and that competition is compatible with this (but also any other) distribution. There is, similarly, an optimal size distribution of wage-setting firms: the typical firm size is larger than in competition. However, wage setting is not compatible with this distribution because firms would like to grow as large as possible. Competitive firms have returns to scale which eventually diminish; wage-setting firms choose wages such that increasing returns persist at the chosen wage.

Related literature: This paper touches on several issues which have been the subject of recent research. For example, we take one view of what a firm is. It is an organization which sets the boundaries of collaboration between workers, and so internalizes productive externalities. This is a view taken implicitly by Holmstrom (1982), and more explicitly by Prescott and Boyd (1987, 1988), among others. It puts the emphasis on technology rather than transactions capabilities as the defining characteristic. Coase (1936) asked what it is that can be done within a firm but not without it; surely, some of the explanation must lie in understanding the characteristics of production itself. After all, we are more willing to join firms to produce goods and services than to consume them. The firm produces goods, and it also produces experience, albeit second-hand for some. The firm owns the goods, but workers own the experience. The inalienability of experience is a recurring theme in transactions costs theories of the firm (Hart, 1989; Moore, 1992) and appears as a transactions constraint for the firm. Ex ante, the firm can make the workers pay for the fact that they will get something of value at some time by working for the firm, rather than outside it.

We have already alluded to models of growth that consider increasing returns arising from the accumulation of knowledge: externalities which arise because of spillovers are hypothesized to generate sustained productivity growth. Our work is related to some other streams of research, on R&D in the industrial organization literature, to ideas of social learning, and to work in organization theory on firm size in the tradition of Williamson (1967). There is a very
extensive literature which deals with interactions among competing firms conducting R&D. Of particular relevance is the manner in which this work extends the search framework (e.g., Weitzman, 1979) to a multi-agent setting (Reinganum, 1983; Bhattacharya et al., 1986).

2. The technology

The production function of the firm is derived from technological possibilities as constrained by the incentives of workers and the mechanisms for learning. At any given time, a worker can be involved in looking for methods to improve productivity ('research'). Research requires effort, but makes discovery possible with some probability. A worker who is not involved in research cannot make a discovery. However, once the efficient method is discovered by at least one worker, all her colleagues can effortlessly imitate it from the next period on – the fruits of research are nonexclusive. The firm internalizes this spillover. This may make a larger firm more productive. At the same time, workers can free ride on the research of others; and the incentives to do so are greater in a larger firm: if many others are engaged in research, the relative benefit of free riding is higher.

Research takes place contemporaneously with production, and there is only one potential productivity improvement to be made. Effort expended on research does not detract from productivity, so it is useful to think of it as the worker giving up leisure. As long as the efficient method is unknown, the worker produces at a lower level of productivity. Effort is unobservable so that wages can only be based upon output.

The specification of learning determines the form of the intertemporal production schedule. As a result, to obtain the derived production function, we need to first describe the output in each state of discovery, as well as the evolution of the state as it results from learning.

2.1. Research and productivity

To motivate the process of research and productivity increase, suppose that each worker is assigned to a task, identical across workers. This task can be performed efficiently – with productivity $\rho_H$ – but this is possible only if the worker knows the best method of doing it; otherwise, the task can still be performed, but less efficiently. We write $\rho_L$ for this lower productivity per unit of labour time. Any worker who once observes the efficient method is able to use it from that point on. Since all individual outputs are observed within the firm, it suffices that one worker find the correct method. All others can then imitate her from the very next period on (which they will do if they are paid more for being efficient).
The utility cost of research effort will be denoted by $\theta$, and the effort taken to work is normalised to zero. The workers who engage in research have probability $\pi$ of finding the efficient method. As we see later, the ratio $\delta = \theta/\pi$ is a natural index for measuring the degree of difficulty of a research project. Clearly, less difficult research will be performed more often and the incentive to free ride is higher for more difficult problems. Knowledge of the best technique is inalienable. Once a worker knows the efficient method, she knows it for ever and will be able to produce $\rho_H$ every period thereafter.

2.2. Production and discovery

The number of high productivity workers in a firm at any point in time is a random variable which describes the state of discovery. In this section we examine the evolution of this state. Production takes place over discrete time $t = 1, 2, \ldots$. The firm starts with a labour force of $N$ workers of whom some number $M$ do research ($M \leq N$). If, at time $t$, $M > 0$, then there is positive probability of discovery in that period. If many workers are involved in research, simultaneous discoveries are possible.

We make the simplifying assumption that $\pi$, the probability of discovery by any one worker, does not change over time and that the outcome of research is independent over time as well as across people. This amounts to research without recall and also implies that imitation is the only source of interaction in research. Further, there is only one discovery to be made after which research is stopped.

Suppose $M$ workers choose to do research in every period. Then the probability that no discovery is made in a particular period is $q_M = (1 - \pi)^M$. The probability that no discoveries have been made until time $t$ is $q_M^t$. Let $P(t, m)$ be the probability that the discovery is made by $m$ workers at time $t$ (and not before). The number of workers making a discovery in any period is a Binomial random variable with parameters $(M, \pi)$, so that

$$P(t, m) = q_M^{t-1} \frac{M!}{m!(M - m)!} \pi^m (1 - \pi)^{M - m}.$$ 

The time at which the discovery is first made in a firm with $M$ research workers is a random variable, $\tilde{s}_M$: 

$$p_M(s) = \text{Prob}(\tilde{s}_M = s) = \sum_{m=1}^{M} P(s, m) = q_M^{s-1} (1 - q_M), \quad s = 1, 2, \ldots.$$ 

The expected time of discovery in this firm is $1/(1 - q_M)$ so that earlier discoveries can be expected either with a large research group or with a research project.
that has a higher probability of success. The number of simultaneous discoveries is also a random variable, $\tilde{m}_M$.

Since labour productivity at any time depends upon the state of discovery, which is a random variable, output is stochastic as well. Let $(s, m)$ be any realization of $(\tilde{s}_M, \tilde{m}_M)$, with $s \in \{1, 2, \ldots \}$ and $m \in \{1, \ldots, M\}$. Conditional on $(s, m)$, output levels are

$$X(t; s, m) = \begin{cases} \rho_L N & \text{for } t < s, \\ \rho_L(N - m) + \rho_H m & \text{for } t = s, \\ \rho_H N & \text{for } t > s. \end{cases}$$

The output of the firm at time $t$ can be written as

$$X(t) = \tilde{\rho}_M(t) N,$$

with $\tilde{\rho}_M(t)$ a random variable for each $t \geq 1$.

We are now in a position to examine first the evolution of the state of discovery and then of the productivity. Labour productivity $\tilde{\rho}_M(t)$ is equal to $[\rho_L(N - n_t) + \rho_H n_t]/N$, where $n_t$ is the number of high productivity workers at time $t$. The state variable $n_t$ is random, takes values in $\{0, 1, 2, \ldots, M\} \cup \{N\}$, and evolves according to a Markov process, which drives the productivity process $\tilde{\rho}_M(t)$, also Markovian. The probability of a state at time $t + 1$ depends only upon the state of discovery at time $t$. Let $M < N$.\(^1\) The transition probabilities for $n_t$ are given by

$$P(n_{t+1} = n \mid n(t) = 0) = \frac{M!}{n!M - n!} \pi^n (1 - \pi)^{M-n} \text{ for } n = 0, 1, \ldots, M,$$

$$P(n_{t+1} = N \mid n(t) = 0) = 0,$$

and

$$P(n_{t+1} = n \mid n(t) > 0) = 0 \text{ for } n = 0, 1, \ldots, M,$$

$$P(n_{t+1} = N \mid n(t) > 0) = 1.$$

If a discovery has not yet been made, then the probability that $n \leq M$ workers will make a discovery follows the Binomial distribution; the probability that all

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\(^1\)For $M = N$, the event $n(t) = N$ is interpreted as the union of $n(t) = M$ and $n(t) = N$.\]
N workers will find the efficient technique is zero. Once a positive number of people know the efficient technique, all N workers will know the efficient technique at all time thereafter. This Markov process has no stationary distributions other than degenerate ones. For $M = 0$, this distribution is degenerate at $n = 0$; for $M > 0$, the only stationary distribution is degenerate at $n = N$.

We want to look at the evolution of productivity starting from the initial condition $n_0 = 0$. The probability distribution of $n(t)$ is

$$P(n(t) = n) = P(t, n) \quad \text{for} \quad n = 1, 2, \ldots, M,$$

$$P(n(t) = N) = 1 - q_M^{-1}.$$

Labour productivity is linear in $n$; it has expectation

$$\bar{\rho}_M(t) = E[\rho(t)] = \rho_H - q_M^{-1}(\rho_H - \rho_L) \left(1 - \frac{\pi M}{N}\right),$$

and variance

$$V_M(t) = (\rho_H - \rho_L)^2 \left( q_M^{-1} \frac{M\pi(1 - \pi)}{N^2} + q_M^{-1} \left(1 - q_M^{-1} \left(1 - \frac{M\pi}{N}\right)^2\right) \right).$$

For each $t$, expected labour productivity is increasing in $M$, the number of workers engaged in research. Research is a source of increasing returns; every worker's productivity increases if any single one of them is successful. The probability of success by at least one increases exponentially with the total number. For each $M > 0$, $\lim_{t \to \infty} \bar{\rho}_M(t) = \rho_H$ and $\lim_{t \to \infty} V_M(t) = 0$; any amount of research, however small, eventually guarantees success. If $M = 0$, labour productivity is both certain and constant in time: $\bar{\rho}_0(t) = \rho_L$ at each $t$.

The number of workers, $M$, who do research has so far been taken as given. However, the firm cannot force any worker to do research or, for that matter, not to do it. They can influence a worker's decision by the wage contract, but the value of $M$ must be determined from the optimizing choice of workers. This is what we turn to now.

3. Workers: The participation constraint

Any wage contract that a firm may offer workers must satisfy constraints imposed by the presence of outside opportunities available to them. Here, self-employment determines the reservation wages and utility levels. This section begins by establishing the parametric conditions for self-employed workers to
do research; the payoff differential from discovery must be large enough to compensate for the difficulty of doing research. The reservation utility is then described as a function of the state of knowledge.

3.1. Utility from self-employment

We will assume, for simplicity, that workers are identical in preferences as well as skills. Each worker maximizes lifetime utility, which depends on consumption as well as effort spent. At any point in time, a typical worker's utility is $U(w) - \theta$ if she gets wage $w$ and is doing research, and $U(w)$ otherwise. Effort does not detract from productivity, so that $\theta$ can be thought of as a leisure cost. $U$ is increasing and concave and represents the instantaneous utility of income. Let $e_t \in \{0, 1\}$ represent the decision to spend research effort, so that instantaneous utility at $t$ is $U(w_t) - \theta e_t$. The lifetime utility of a worker depends on the profiles of wages, $w = \{w_t; t = 1, 2, \ldots \}$, as well as effort choices, $e = \{e_t; t = 1, 2, \ldots \}$,

$$V(w; e) = (1 - \beta) \sum_{t=1}^{\infty} \beta^{t-1} (U(w_t) - \theta e_t), \quad e_t \in \{0, 1\}.$$  

The discount factor $\beta \in (0, 1]$ is common to all workers. In situations of uncertainty, they use the expected utility criterion. Workers may choose to stop or start research at any point by choosing $e_t$ at $t$. Each worker is endowed with one unit of labour each period.

Self-employment is an option open to every worker in each period and is used to determine the reservation wages and utility levels. It turns out that a worker will either choose to do research in the first period and continue until successful or else will never do research at all. The following lemma sets out the parametric conditions for self-employed workers to do research.

**Lemma 1.** Self-employed workers will do research if and only if

$$U(\rho_H) - U(\rho_L) \geq (1 - \beta) \frac{\theta}{\pi}.$$  

**Proof.** Let $\tilde{s}_1$ be the time of discovery by a self-employed worker. Consider any $t \leq \tilde{s}_1$. Let $V_*(e_t)$ be the expected lifetime utility of a self-employed worker who has chosen $e_t$ at this time. If $e_t = 1$, the worker spends effort $\theta$ at $t$, whatever the outcome. With probability $\pi$, she makes a discovery and is assured of income $\rho_H$ from $t$ on; failing that, she earns $\rho_L$ and can choose $e$ again next period. It follows that

$$V_*(1) = \pi U(\rho_H) + (1 - \pi)((1 - \beta) U(\rho_L) + \beta \max_e V_*(e)) - \theta(1 - \beta).$$  


If $e_t = 0$, she spends zero effort at $t$, has income $\rho_L$, and can choose $e$ again next period, so that

$$V_*(0) = (1 - \beta)U_\rho + \beta \max_e V_*(e).$$

A self-employed worker will choose to do research at each $t < \bar{t}$ if and only if $V_*(1) \geq V_*(0)$. Suppose $V_*(1) \geq V_*(0)$. This implies $V_*(1) = \max_e V_*(e)$, so that

$$V_*(1) = \frac{\pi U_\rho + (1 - \pi) (1 - \beta)U_\rho - \theta(1 - \beta)}{1 - \beta(1 - \pi)},$$

$$V_*(0) = (1 - \beta)U_\rho + \beta V_*(1).$$

By substitution,

$$V_*(1) \geq V_*(0) \implies U_\rho - U_\rho \geq (1 - \beta) \frac{\theta}{\pi}. $$

If, on the contrary, $V_*(1) < V_*(0)$, we have $V_*(0) = U_\rho$ and

$$V_*(1) = \pi U_\rho + (1 - \pi)U_\rho - \theta(1 - \beta).$$

It follows that

$$V_*(1) < V_*(0) \implies U_\rho - U_\rho < (1 - \beta) \frac{\theta}{\pi}. $$

This proves the result. 

The quantity $\Delta = U_\rho - U_\rho$ is the differential payoff of successful research; while $\delta = \theta/\pi$ measures the inherent difficulty of research. The condition

$$\Delta \geq \delta(1 - \beta)$$

requires that the payoff be large enough to compensate for the difficulty. Failing this, research is unattractive, and the worker will choose $e_t = 0$ each period. Her income is $\rho_L$ at each $t$. Any single worker will always do research if she does not discount the future, i.e., $\beta = 1$.

Workers are free to quit at any time. Since self-employment is a feasible option for all workers, it determines their reservation utility levels. The
participation constraint of each worker is sequential. At any time after the worker learns the efficient technique her wage must be at least $\rho_H$ (regardless of whether she made the discovery herself or by observation). At any time before learning $\rho_H$, the reservation utility level is $\max(V_*(0), V_*(1))$. The next lemma summarizes this.

**Lemma 2.** Let $\tilde{s}$ be the time at which a worker learns $\rho_H$. The reservation utility level for this worker is

$$U_* = U(\rho_L) + \max \left( \frac{\pi(U(\rho_H) - U(\rho_L)) - (1 - \beta)\delta}{1 - \beta(1 - \pi)}, 0 \right) \quad \text{for} \quad t < \hat{s},$$

$$U_* = U(\rho_H) \quad \text{for} \quad t \geq \hat{s}.$$

**Proof.** Note that $V_*(1) \geq V_*(0)$ implies

$$V_*(1) = U(\rho_L) + \frac{\pi(U(\rho_H) - U(\rho_L)) - (1 - \beta)\delta}{1 - \beta(1 - \pi)},$$

and $V_*(1) < V_*(0)$ implies $V_*(0) = U(\rho_L)$. For $t < \hat{s}$, the worker’s reservation utility is equal to the higher of the two; by the previous lemma, $V_*(1) \geq V_*(0)$ if and only if $\Delta > (1 - \beta)\delta$. For $t \geq \hat{s}$ self-employment guarantees $w_{t+k} = \rho_H$ and $e_{t+k} = 0$ for each $k \geq 0$. $\Box$

4. Incentives and research in the firm

If a worker is in a firm, the presence of other researchers increases the rewards to shirking, since she can count on the success of others to enhance her own productivity and income. As a result workers have a greater propensity to shirk, and the larger the firm, the greater the incentive to free ride in this manner. These strategic opportunities and the interaction can be modelled explicitly as a game. We will do this and, from the Nash equilibrium conditions, deduce the worker incentive compatibility constraints.

The firm can hope to offset free riding incentives with the choice of a suitable wage contract. However, it is not entirely free to choose $w_i$, since workers must be willing to work for those wages. In this section, we explore the constraints set by the fact that research must be incentive-compatible. The individual’s choice depends on the wage schedule, $w$, and the number of research workers in the firm, $M$. A firm can have $M$ researchers if exactly $M$ workers choose to do research.
4.1. The research decision

We will describe the game played by \( N \) workers, taking the wage contract as given. This contract is chosen by the firm which assumes that workers will then use Nash equilibrium strategies. The firm chooses this contract and \( M \) to maximize profits subject to the participation and incentive compatibility constraints obtained from the workers’ game.

Consider a firm where \( M - 1 \) workers choose to do research; we examine the decision of an additional worker – the \( M \)th. Since workers are identical, each research worker goes through an identical thought experiment. It will be incentive-compatible for \( M \) workers to do research if (i) for every worker who is required to do research, this decision is optimal given that \( M - 1 \) do research, and (ii) for workers not required to do research, this is optimal given that \( M \) do research. If the equilibrium were unique, then workers would be able (in principle) to compute the equilibrium of the game and determine their own action. Here we have multiple equilibria (as many as there are \( M \)-sized subcollections of \( N \)). We proceed by assuming that the firm can choose any one of these equilibria. Once equilibrium actions have been prescribed, it will be in each worker’s interests to follow instructions. However, in general, a worker will prefer an equilibrium in which she is not required to do research.

Firms cannot, or find it very costly to, monitor efforts. It is also possible that difficulties in the measurement of effort\(^2\) might make it difficult to enforce contracts contingent upon it. For such reasons we preclude the possibility of contracting on effort. \( \text{Ex post} \), the firm can identify successful researchers on the basis of their productivity. All workers can identify and imitate successful researchers with a one-period lag. Workers and firms are assumed not to observe what is happening in other firms, and we also rule out migration by workers to other firms or ‘corporate raiding’ by firms.

The form we assume wage contracts to take is \((w_L, w_H)\), and contingent only upon the individual worker’s productivity. Should a firm (which is not a price-taker) reward people identified (\( \text{ex post} \)) as researchers with payments in excess of their (now) higher productivity? We have noted the public good nature of knowledge and observed that contracts must be such that, in every time period, it is in the interests of all parties to abide by it (specifically, promises of rewards to successful researchers must be credible). Later in the paper we see that even if firms could credibly commit to reward discovery, it may not be in their interests to do so. A source of profits for the firm (here the only one) is the fact that it can make workers pay for the valuable experience they gain. Profits are made off workers who don’t know the better technique and early discovery makes the profitable period too short. The optimal contract balances the gains from

\(^2\)It could be possible to look busy without doing any real work.
extending the profitable period with the cost of having to offer a larger \( w_L \) when this period is longer.

The next lemma establishes the conditions under which a worker will join \( M - 1 \) others in doing research. We consider the expected utility, \( V_M(e; w) \), of a worker who chooses \( e \in \{0, 1\} \), and faces a wage schedule \( w = (w_H, w_L) \). At this time, one of three mutually exclusive events occurs: the worker makes a discovery \((E_1)\); she does not, but at least one other worker does \((E_2)\); and none do \((E_3)\). For an (active) researcher their probabilities are \( \text{Prob}(E_1) = \pi \), \( \text{Prob}(E_3) = q_M = (1 - \pi)^M \), \( \text{Prob}(E_2) = (1 - \pi - q_M) \). A worker who chooses \( e = 0 \) can only be in states \( E_2 \) or \( E_3 \), with \( \text{Prob}(E_2) = (1 - q_{M-1}) \) and \( \text{Prob}(E_3) = q_{M-1} \). A worker will choose to do research if, and only if \( V_M(1; w) \geq V_M(0; w) \). The next result obtains by applying sequential (or 'temporary') incentive compatibility conditions as in Green (1987).

**Lemma 3.** Facing a wage schedule \( w = (w_L, w_H) \), a worker will choose to join \( M - 1 \) others in doing research if and only if

\[
U(w_H) - U(w_L) \geq (1 - \beta q_{M-1}) \frac{\theta}{\pi},
\]

where \( q_{M-1} = (1 - \pi)^{M-1} \).

**Proof:** Let \( \hat{s}_M \) be the time at which discovery occurs and \( V_M(e, w|E_i) \) the utility of a worker who has chosen \( e \) at \( t \leq \hat{s}_M \) conditional on event \( E_i, i = 1, 2, 3 \). We have

\[
V_M(1, w | E_1) = (1 - \beta)(U(w_H) - \theta) + \beta U(w_H),
\]

\[
V_M(1, w | E_2) = (1 - \beta)(U(w_L) - \theta) + \beta U(w_H),
\]

\[
V_M(1, w | E_3) = (1 - \beta)(U(w_L) - \theta) + \beta \max_e V_M(e, w),
\]

and

\[
V_M(1, w) = \pi V_M(1; E_1) + (1 - \pi - q_M) V_M(1; E_2) + q_M V_M(1; E_3).
\]

Similarly,

\[
V_M(0, w | E_2) = (1 - \beta) U(w_L) + \beta U(w_H),
\]

\[
V_M(0, w | E_3) = (1 - \beta) U(w_L) + \beta \max_e V_M(e, w).
\]
and

\[ V_M(0, w) = (1 - q_{M-1}) V_M(0; E_L) + q_{M-1} V_M(0; E_\lambda). \]

The worker chooses to do research if and only if \( V_M(1) \geq V_M(0) \). Suppose \( V_M(1, w) \geq V_M(0, w) \); then \( V_M(1, w) = \max_e V_M(e, w) \), and

\[ V_M(1, w) = (1 - \lambda_M) U(w_H) + \lambda_M U(w_L) - \frac{\theta}{1 - \pi} \lambda_M, \]

where \( \lambda_M = (1 - \pi)(1 - \beta)/(1 - \beta q_M) \), and

\[ V_M(0, w) = \beta(1 - q_{M-1}) U(w_H) + (1 - \beta) U(w_L) + \beta V_M(1, w). \]

Using the fact that \( q_M = q_{M-1}(1 - \pi) \), substitutions yield

\[ V_M(1) \geq V_M(0) \quad \Rightarrow \quad U(w_H) - U(w_L) \geq (1 - \beta q_{M-1}) \frac{\theta}{\pi}. \]

If \( V_M(1) < V_M(0) \), this implies \( V_M(0, w) = \max_e V_M(e, w) \), and

\[ V_M(1) = (\pi + \beta(1 - \pi)(1 - q_{M-1})) U(w_H) + (1 - \beta) U(w_L) - (1 - \beta q_{M-1}) V_M(0), \]

\[ V_M(0) = \frac{(1 - \beta) U(w_L) + \beta(1 - q_{M-1}) U(w_H)}{1 - \beta q_{M-1}}. \]

Substitutions yield

\[ V_M(1) < V_M(0) \quad \Rightarrow \quad U(w_H) - U(w_L) (1 - \beta q_{M-1}) \frac{\theta}{\pi}. \]

This proves the result. □

In large firms, where many workers are already engaged in research, a single worker will have more of an incentive to shirk. The quantity \( 1 - \beta q_{M-1} \) is increasing in \( M \). Free riding is more attractive if there is a larger pool to benefit from.
4.2. Wage differentials and research

If the payoff differential is greater than \((1 - \beta)\delta\), some workers will do research; if it is sufficiently high, all workers will, irrespective of the size of the firm. For intermediate values some do research while others shirk.

**Lemma 4.** Let \(N > 0\) be the number of workers in a firm with wage schedule \(w = (w_H, w_L)\).

1. \(M = 0\) if and only if \(U(w_H) - U(w_L) < (1 - \beta)\theta/\pi\).
2. \(M < N\) if and only if \(U(w_H) - U(w_L) < (1 - \beta q_{N-1})\theta/\pi\); \(M\) is such that \((1 - \beta q_{M-1})\theta/\pi > U(w_H) - U(w_L) \geq (1 - \beta q_{M-1})\theta/\pi\).
3. \(M = N\) if and only if \(U(w_H) - U(w_L) \geq (1 - \beta q_{N-1})\theta/\pi\); this is true for all \(N\) if \(U(w_H) - U(w_L) > \theta/\pi\).

**Proof.** This is a direct application of Lemma 3. The first statement follows from the fact that \((1 - \beta q_{M-1}) \leq (1 - \beta)\) if \(M > 0\). If \(\Delta(w) \geq (1 - \beta)\delta\), at least one worker will do research. If \(M < N\), it must be that \(V_M(1, w) \geq V_M(0, w)\) and \(V_{M-1}(1, w) < V_{M-1}(0, w)\). This implies the second statement. Finally, \(q_N > 0\), so that \(M = N\) for all \(N\) is true whenever \(V_\alpha(1, w) \geq V_\alpha(0, w)\). \(\Box\)

We now define a function, \(\mu(w)\), which will be very useful in the subsequent analysis:

\[
\mu(w) = \begin{cases} 
0 & \text{if } \Delta(w) < (1 - \beta)\delta, \\
1 + \frac{\ln(\beta \delta) - \ln(\delta - \Delta(w))}{\ln(1/(1 - \pi))} & \text{if } \delta > \Delta(w) \geq (1 - \beta)\delta, \\
\infty & \text{if } \Delta(w) > \delta,
\end{cases}
\]

with \(\delta = \theta/\pi\) and \(\Delta(w) = U(w_H) - U(w_L)\), as usual.

For each \(w\), the number of workers doing research is

\[M = \min(\lfloor \mu(w) \rfloor, N),\]

where \(\lfloor y \rfloor\) is the largest natural number less than the argument \(y\). For \(\Delta(w)\) sufficiently small, \(\lfloor \mu(w) \rfloor = 0 = M\), and for \(\Delta(w)\) large enough, \(\mu(w) = \infty\), and \(M = N\) for all \(N\). The partitioning of the parameter space corresponds to zero, finite, and infinite \(\mu\), respectively.
The quantity \( M \) is determined by \( w \), the wage schedule. The value of \( w \) depends on the behaviour of firms, to which we turn in the next section. Before that, it is useful to note that the participation and incentive compatibility constraints of workers imply that \( M > 0 \) is true only if technological parameters satisfy some restrictions.

The wage profile which the firm can actually offer is restricted by the fact that workers must be willing to work for the firm. This leads to \( w_H = \rho_H \), since the reservation wage of a worker who knows the more efficient technique is \( \rho_H \) (and the firm cannot credibly offer \( w_H \) greater than \( \rho_H \)). Next, the wage schedule must be such that \( \max_e V_M(e, w) \geq V_* \), the reservation utility level of workers which puts a lower bound on the entry wage, \( w_L \). \( M > 0 \) is then feasible only if the maximal differential allows for strictly positive \( \mu \).

**Lemma 5.** A firm has \( M > 0 \) workers engaged in research only if

\[
U(\rho_H) - U(\rho_L) \geq (1 - \beta) \frac{\theta}{\pi}.
\]

**Proof.** Let \( w_H = \rho_H \). Then, \( A(w) = U(w_H) - U(w_L) = U(\rho_H) - U(w_L) \). Suppose \( M > 0 \). The utility of a research worker in the firm is

\[
V_M(1, w) = U(\rho_H) - \hat{\lambda}(q_M) \left( A(w) + \frac{\theta}{1 - \pi} \right),
\]

for each \( t < \tau_M \), where \( \hat{\lambda}(q_M) = (1 - \pi) (1 - \beta)/(1 - \beta q_M) \). Sequential participation implies that \( V_M(1, w) \geq V_* \) and this implies, in turn,

\[
A(w) \leq \frac{(1 - \beta q_M)}{(1 - \pi) (1 - \beta)} (U(\rho_H) - V_*) - \frac{\theta}{1 - \pi} \equiv D(q_M, V_*).
\]

\( M > 0 \) only if \( V_M(1, w) \geq V_M(0, w) \). By Lemma 3, this is true only if

\[
D(q_M, V_*) \geq A(w) \geq \delta(1 - \beta q_M)
\]

for some \( M > 0 \). Substitutions yield

\[
D(q_M, V_*) \geq \delta(1 - \beta) \iff (U(\rho_H) - V_* - \theta \frac{(1 - \pi) (1 - \beta)}{\pi}) (1 - \beta q_M) > \frac{\theta}{1 - \pi}.
\]
The right-hand side of the last inequality is decreasing in $q_M$, which is bounded below by 0. This yields the implication

$$U(q_H) - V_* \geq \delta(1 - \beta).$$

The result obtains from the observation that $V_* > U(q_L)$. □

The self-employed do research only if $\Delta \geq (1 - \beta)\delta$. The last lemma establishes a stronger result: this condition is necessary for any research to be done.

5. The firm: Objectives and outcomes

The productivity of labour in a firm depends on how much research takes place within it and this is determined by the wage differential in the firm. The objectives and behaviour of the firm determine wages and employment and, as a consequence, the production possibilities within the firm and in the aggregate.

We examine three kinds of firms: a workers' cooperative, a competitive firm, and a wage-setting firm. In the cooperative workers retain the rights to their own output, the firm simply offering them the opportunity to work together. The competitive firm acts as a price-taker in the labour market. The wage-setting firm chooses wages to maximize profits given the participation and incentive constraints.

We begin this section with a characterization of production and profits for different wage-employment levels. This involves combining the incentive and participation constraints with the results of Section 2.

5.1. Output and profits

To analyze the effects of firm behaviour on output we look at the expected present value of the output stream at $t = 0$. Firms are assumed to maximize expected profits, also discounted by the common factor $\beta$.

Consider a firm with $N$ workers, of whom $M$ do research. At time $t \geq 1$, the random output is $\tilde{\rho}_M(t)N$. Let $\tilde{X}(M, N)$ be the expected, discounted present value of output at $t = 1$:

$$\tilde{X}(M, N) = (1 - \beta) \sum_{t=1}^{\infty} \beta^{t-1} E\tilde{\rho}_M(t)N.$$  

Using, from Section 2, the fact that $\tilde{\rho}_M(t) = \rho_H - q^{-1}_M(\rho_H - \rho_L)(1 - M\pi/N)$, we have

$$\tilde{X}(M, N) = \phi(M, N)N,$$
where

\[ \phi(M, N) = \rho_H - \frac{(1 - \beta)}{1 - \beta q_M (\rho_H - \rho_L)} \left( 1 - \frac{M \pi}{N} \right). \]

\( \phi(M, N) \) is the present value of labour's marginal product, which is strictly increasing and concave in \( M \) and nonincreasing in \( N \). The first property is intuitive and indicates that research is productive, with diminishing returns. The second is due to the fact that an increase in \( N \), with \( M \) constant, decreases the proportion doing research. Lemma 4 gives the number of researchers at a given wage profile: \( M = \min(\mu(w), N) \). If a firm has fewer than \( \lfloor \mu \rfloor \) workers, they will all be willing to do research.

We can write output as a function of employment, with \( \mu \) as a parameter:

\[ X(N) = \phi_\mu(N)N = \max(\phi(\mu, N), \phi(N, N))N. \]

For the case where \( \mu = \infty \), define

\[ \phi_x(N) = \rho_H - \frac{(1 - \beta)(1 - \pi)}{1 - \beta q_N (\rho_H - \rho_L)} \]

and denote the production function by

\[ X_x(N) = \phi_x(N)N. \]

Lemma 6. Let \( \phi_\mu(N) = \phi(M, N) \) evaluated at \( M = \min(N, \mu) \).

1. \( \mu = 0 \Rightarrow \phi_0(N) = \rho_L \) for all \( N > 0 \).
2. \( 0 < \mu < \infty \Rightarrow \phi_\mu \) is increasing in \( N \) when \( N < \lfloor \mu \rfloor \), reaches a maximum at \( N = \lfloor \mu \rfloor \), and decreases thereafter. Further, for \( N > 0 \),

\[ \phi_\mu \leq \rho_H - (1 - \pi) \frac{1 - \beta}{1 - \beta q_\mu (\rho_H - \rho_L)}. \]

and for \( N \geq \lfloor \mu \rfloor \),

\[ \phi_\mu > \lim_{N \to \infty} \phi_\mu(N) = \rho_H - \frac{1 - \beta}{1 - \beta q_\mu (\rho_H - \rho_L)}. \]

3. \( \mu = \infty \Rightarrow \phi_x \) is increasing and concave in \( N \). Further,

\[ \phi_x < \lim_{N \to \infty} \phi_x(N) = \rho_H(1 - \beta) + \rho_L(1 - \pi)(1 - \beta). \]
Proof: Statement (1) is obvious. To prove the other two, note first that $M = \lfloor \mu \rfloor$ for $N > \mu$, and $M = N$ otherwise. Also, from Lemma 3, if a worker will choose to join $M - 1$ others in doing research at a given $w$, so will she choose to join $M - 2$ others. For $N \leq \lfloor \mu \rfloor$, per capita productivity is equal to $\phi(N, N)$, which is increasing and concave in $N$. To see this, let

$$\phi(N) = \phi(N, N) = \rho_H - (1 - \beta) \frac{(1 - \pi)}{(1 - \beta q_N) (\rho_H - \rho_L)},$$

which with $q_N = (1 - \pi)^N$ implies

$$\phi(N + 1) - \phi(N) = \frac{\beta(1 - \beta)\pi (1 - \pi)^N}{(1 - \beta q_N) (1 - \beta q_{N+1})} (\rho_H - \rho_L) > 0.$$

Concavity follows from the fact that

$$\phi(N) + \phi(N - 2) < 2\phi(N - 1),$$

whenever $(1 - \beta q_{N-1}) > 0$.

For all $N > \mu$, per capita productivity is

$$\phi_\mu(N) = \rho_H - (\rho_H - \rho_L) \frac{1 - \beta}{(1 - \beta q_\mu)} \left(1 - \frac{\lfloor \mu \rfloor \pi}{N}\right),$$

which is strictly decreasing in $N$.

This proves assertions 2 and 3, noting that $\phi_\mu(N) \leq \phi_\mu(\lfloor \mu \rfloor)$ and that $\phi_\mu$ asymptotes to its limit from above when $\mu < \infty$ and from below otherwise. □

If $\mu = 0$, average and marginal productivities are constant and equal to $\rho_L$: no worker does research and the low-productivity method is used. If $0 < \mu < \infty$, in a small firm, new workers do research and contribute to average productivity. Once the firm has reached its critical size, every new worker is a shirker and contributes exactly the same as every other. This is positive, but generates no spillovers. Shirkers have lower expected productivity, so that the average falls as their number increases. If $\mu = \infty$ (research is incentive compatible for all workers), the production function is S-shaped, with increasing returns at low levels of employment, and diminishing returns thereafter as the next lemma establishes.

Lemma 7. The function $X_\omega(N) = \phi_\omega(N)N$ is increasing, convex at $N = 0$, and concave for $N > N^*$, with $0 < N^* < \infty$. 
Given $X_\infty(N) = \phi_\infty(N) N$, where

$$\phi_\infty(N) = \rho_H - \frac{(1 - \beta)(1 - \pi)}{1 - \beta q_N}(\rho_H - \rho_L),$$

we have

$$X_\infty(N + 1) - X_\infty(N) = N(\phi_\infty(N + 1) - \phi_\infty(N)) + \phi_\infty(N + 1)$$

$$= \frac{(1 - \beta)(1 - \pi)(\rho_H - \rho_L)\beta \pi q_N}{(1 - \beta q_N)(1 - \beta q_{N+1})} > 0.$$

The sign of $X_\infty(N) + X_\infty(N - 2) - 2X_\infty(N - 1)$ is the same as that of $a(N)N + b(N)$, where we always have $a(N) < 0$ and $b(N) > 0$. For small values of $N$, this quantity $a(N)N + b(N) > 0$, so that the function $X_\infty$ is convex; as $N$ is increased, $a(N)N + b(N) < 0$ and $X_\infty$ becomes concave.

Recall that $\mu$ is determined by the wage schedule. If the wage differential, $w_H - w_L$, is sufficiently low, the production function is of an 'efficiency wage' form and labour productivity is determined by wages. There is a difference with the usual version of the efficiency wage hypothesis. If $w_L = \rho_L$ (by sequential participation), labour productivity is decreasing in $w_L$ so that firms would need to lower entry wages to increase labour quality. The effect of wages on productivity is limited. For a sufficiently steep wage profile, labour productivity is $\phi(N)$ and the firm has increasing returns to scale.

While the production function is eventually concave, the profit function need not be. If firms set wages, larger firms can pay lower entry wages and thereby cut costs per worker. Since workers would like to learn higher productivity techniques earlier, this promise can be credibly made by a large firm. If workers prefer to work for a large firm, the firm can exploit this by offering them slightly lower wages. The ability of larger firms to attract workers at lower entry wages makes it possible for profits to be always increasing in firm size even though the production function displays diminishing returns. Expected discounted profits can be shown to be

$$\Pi(w_L, M, N) = (\rho_L - w_L)(1 - \beta)\frac{N - (1 - q_M)\pi M}{1 - \beta q_M}.$$

A firm with $N > 0$ must have $w_L \leq \rho_L$. For each $w_L < \rho_L$, profits are strictly increasing in $N$ and decreasing in $M$. 
5.2. The firm as a workers' cooperative

In a workers' cooperative of size $N$ every worker has the right to her own output and effectively faces a wage schedule $\rho = (\rho_H, \rho_L)$. The benefit to workers, of working in the same firm, comes purely from the possibility of learning from each other (and is positive when $M > 0$). Further, $M = \max(\mu(\rho), N)$.

**Proposition 1.** A workers' cooperative pays $w_H = \rho_H, w_L = \rho_L$.

1. If $U(\rho_H) - U(\rho_L) < (1 - \beta)\theta/\pi$, labour productivity is $\rho_L$ in every cooperative, irrespective of size.

2. If $\theta/\pi > U(\rho_H) - U(\rho_L) \geq (1 - \beta)\theta/\pi$, labour cooperatives have a minimum efficient scale, $\bar{N} = \mu(\rho)$. Labour productivity is maximal at $N = \bar{N}$.

3. If $U(\rho_H) - U(\rho_L) > \theta/\pi$, labour productivity increases with the size of the cooperative, and converges to a positive constant from below.

**Proof.** This result is a direct application of Lemma 6 and Lemma 7, after noting that

\[
\mu(\rho) = \begin{cases} 
0 & \text{if } \Delta < (1 - \beta)d, \\
0 < \mu(\rho) < \infty & \text{if } \delta > \Delta \geq (1 - \beta)d, \\
\mu(\rho) = \infty & \text{if } \Delta \geq \delta.
\end{cases}
\]

For each $\rho, M = \min(\mu(\rho), N)$ and $\phi = \phi_{\mu(\rho)}$ are fully determined. \(\square\)

5.3. Competition and firm size

Consider next an economy where firms are competitive, taking wages as given. The wage $w_H$ must be equal to $\rho_H$ as before. There remains the entry wage, $w_L$. The profits of the firm are of the form

\[
\Pi(w_L, M, N) - (\rho_L - w_L)F(M, N),
\]

$F$ and $\partial F/\partial N$ are both strictly positive. For $w_L > \rho_L$, firms would not hire, and if $w_L < \rho_L$, competitive firms would make positive profits per worker and demand infinite amounts of labour. It follows that $w_L = \rho_L$ in a competitive labour market – at equilibrium, competitive firms are exactly like worker's cooperatives. Competitive firms make zero profits despite the possibility of increasing returns to scale. The usual justification for competition relies on there being a large number of small firms. In the present model, firms make zero
profits at every size and are quite indifferent about \( L \). This is compatible with the existence of a large number of small firms, even if it does not imply it.

The size and number of firms is not irrelevant to society as a whole; with increasing returns to labour in production, aggregate production possibilities depend critically on the size of firms. Suppose that \( \bar{N} \) is the total amount of labour available, and further, that firms are competitive, so that \( w = p \) at equilibrium. How large should firms be? If \( X(N) \) is the production function of a single firm, the aggregate production function is

\[
X(N_1, \ldots, N_K) = \sum_{i=1}^{K} X(N_i), \quad \sum_{i=1}^{K} N_i \leq \bar{N},
\]

with \( K \) firms. What is the size distribution of firms necessary for production efficiency?

Proposition 2. Let \( \bar{N} \) be the aggregate supply of labour.

1. If \( U(\rho_H) - U(\rho_L) < (1 - \beta)\theta/\pi \), aggregate output is invariant to the size distribution of competitive firms.
2. If \( \theta/\pi > U(\rho_H) - U(\rho_L) \geq (1 - \beta)\theta/\pi \), and \( \bar{N} \) sufficiently large, the optimal size of competitive firms is \( \left\lfloor \mu(\rho) \right\rfloor \).
3. If \( U(\rho_H) - U(\rho_L) > \theta/\pi \), the optimal size of a competitive firm is \( \bar{N} \).

Proof. The first statement is obvious, as \( X(N) = \rho_L N \) for all firms.

If \( \delta > A \geq (1 - \beta)\delta \), \( \mu(\rho) \) is finite and positive, by Proposition 1. Further, average productivity attains its maximum at \( \mu(\rho) \). Suppose that all firms are of the same size. Then \( X(\bar{N}) = KX(N/K) \), and \( \max_{K} X(N) \) is equivalent to maximizing average productivity \( X(\ell)/\ell \) for each \( \bar{N} \). This maximum is attained at \( \ell = \left\lfloor \mu(\rho) \right\rfloor \). If firms are of different sizes, aggregate output can be increased by reallocation.

If \( A \geq \delta \), \( \mu(\rho) = \infty \), and average productivity is strictly increasing in firm size. Aggregate output is maximized if all workers are in the same firm. \( \Box \)

If any research is done at the competitive wage, it is optimal to have all workers do research, because of its positive externality. This can be ensured if firms are exactly the right size to have no free riders. If research is easy all workers do research and should belong to the same firm. This may seem counterintuitive, but recall that the bound is set by incentive considerations.
5.4. Wage setting and monopsony

Profits and productivity depend on wages as well as employment. We look now at the firm's choice of wages and the consequent choice of the level of research. First observe that, since incentive and participation constraints must be satisfied, \( M \) and \( w_L \) cannot be determined independently. The following lemma identifies the feasible set \((w_L, M)\) available to the firm. We know that, if \( M = 0 \), then \( w_L = \rho_L \), by the participation constraint. The incentive constraint is meaningful only if \( M \) is positive.

Lemma 8. Let \( w_L \) be the entry wage, and \( M > 0 \) the number of workers doing research. Then \( w_L, M \) satisfy

\[
\alpha - \sigma(1 - \beta q_M) \geq U(w_L) \geq \alpha - \sigma'(1 - \beta q_M),
\]

where \( \alpha = U(\rho_H) + \theta/(1 - \pi) \), \( \sigma = \theta/(\pi(1 - \pi)) \), and \( \sigma' = [U(\rho_H) - U(\rho_L) + \theta/(1 - \pi)]/(1 - \beta(1 - \pi)) \).

Proof. Suppose \( M > 0 \). Then, the incentive constraint implies

\[
U(\rho_H) - U(w_L) \geq (1 - \beta q_{M-1}) \frac{\theta}{\pi},
\]

and the participation constraint implies

\[
U(\rho_H) - U(w_L) + \frac{\theta}{1 - \pi} \leq \frac{(1 - \beta q_M)}{1 - \beta(1 - \pi)} \left( \Delta + \frac{\theta}{1 - \pi} \right).
\]

Write \( X = U(w_L) \) and \( Y = (1 - \beta q_M) \). Note that \( 1 - \beta q_{M-1} = (Y - \pi)/(1 - \pi) \). Substitution yields

\( (IC) \): \( X + \sigma Y \leq \alpha \),

\( (PC) \): \( X + \sigma' Y \geq \alpha \).

This proves the result. \( \square \)

The feasible set is nonempty whenever \( \sigma \leq \sigma' \), which is equivalent to \( \Delta \geq (1 - \beta) \delta \). This is the familiar condition which ensures that the self-employed do research. As it happens, this condition is also necessary for firms to make positive profits.
For the wage-setting firm, there are three possibilities to be considered. The first, where $\Delta < \delta (1 - \beta)$, is uninteresting since a firm cannot get workers at any wage lower than $\rho_L$, and makes no profits (in this case $\mu(\rho) = 0$). The other situations involve $\mu(\rho)$ infinite and finite respectively. These two situations need slightly different types of argument, even though the major qualitative results are similar.

We consider the case $\Delta \geq \delta$ where $\mu(\rho) = \infty$. At $w_L < \rho_L$, all participating workers will choose to do research. Given that $M = N$, the firm will choose the lowest wage at which workers will agree to work: so that the participation constraint binds at the firm's preferred wage solution. Profits will be increasing in $N$. In this case, the firm is assumed to use all of the available labour supply, $N$. [We will argue below that it is also appropriate to interpret $\bar{N}$ as the capacity constraint of a wage-setting firm (which need not be a monopsonist).]

**Proposition 3.** If $U(\rho_H) - U(\rho_L) \geq \theta/\pi$, then $M = \bar{N}$ and a wage-setting firm chooses $w_* < \rho_L$ such that

$$U(w_*) = \alpha - \sigma'(1 - \beta q_N).$$

**Proof.** If $\Delta \geq \delta$, $M = N$ in any firm paying $w_L \leq \rho_L$. From Lemma 8, the incentive constraint with $M = N$ can be written as

$$U(w_L) \geq \alpha - \sigma(1 - \beta q_N),$$

and the participation constraint as

$$U(w_L) \geq \alpha - \sigma'(1 - \beta q_N).$$

Also $\sigma < \sigma'$ whenever $\Delta > (1 - \beta)\delta$. Any $(N, w_L)$ which satisfies the participation constraint with equality also satisfies the incentive constraint.

Profits are $\Pi(w_L, N) = (\rho_L - w_L)F(N)$, $F$ is strictly positive and increasing in $N$, and $\Pi$ is decreasing in $w_L$. By the participation constraint, the lowest wage compatible with any level of employment $N > 0$ is such that $U(w_L) = \alpha - \sigma'(1 - \beta q_N)$. Call this $w_{PC}(N)$, which is strictly decreasing in $N$. For any $N$, this is the profit-maximizing wage, which is less than $\rho_L$ if $N > 1$. Profits are strictly increasing in $N$, and a monopsonist would hire $\bar{N}$ at wage $w_{PC}(\bar{N})$. □

The final case involves a finite $\mu(\rho)$. Here either the participation constraint (PC) binds or the incentive constraint (IC) binds. If PC binds, we have results similar to Proposition 3. If IC binds, wages are higher than reservation wages (an efficiency wage like phenomenon). While $N = \bar{N}$, we have $M < N$, and the firm is too large for productive efficiency.
If $0 < \mu(p) < \infty$, the number of workers who choose to do research will depend on $\Delta(w)$, the wage schedule offered by the firm. Suppose the firm is able to hire $N$ workers. If it chooses an entry wage $w_L$ such that

$$\Delta(w) = U(\rho_H) - U(w_L) \geq \delta(1 - \beta q_{N-1}),$$

all of its work force will choose to do research. This puts the firm in the same position as before: it will choose the lowest possible wage, that is, the participation constraint must bind. For each $N$, this is the wage $w$ such that

$$U(w) = \alpha - \sigma'(1 - \beta q_N).$$

If $w_L$ is such that $\Delta(w) < \delta(1 - \beta q_{N-1})$, some workers out of $N$ will choose to shirk. The number of workers must be such that $M = \mu(\rho_H, w_L)$, so that the incentive constraint binds.

The firm chooses a corner solution $(w_L = w; M = N)$ if and only if it is more profitable than an interior solution $(w_L > w; M = \mu(\rho_H, w_L))$. Since these solution have rather different properties, it is useful to know when one or the other is chosen. The next two results provide a partial answer.

**Proposition 4.** If $\theta/\pi > U(\rho_H) - U(\rho_L) \geq (1 - \beta) (\theta/\pi)$ and $\lim_{w \to \rho} U'(w) = \infty$, a wage-setting firm chooses a wage $w^* \in (0, \rho_L)$ to maximize profits. At this wage, profits are strictly increasing in $N$, so that $N = N$.

**Proof.** For each $w_L < \rho_L$, $M$ is positive, and determined by the incentive constraint:

$$(1 - \beta q_M)\delta > U(\rho_H) - U(w_L) \geq (1 - \beta q_{M-1})\delta.$$ 

This determines $M = \mu(w)$, which is strictly greater than $\mu(\rho)$ whenever $w_L < \rho_L$. Let $w^* = \arg\max_w G(w, N) = (\rho_L - w_L)F(\mu(w), N)$. Maximal profits are attained at $w = w^*$, $M = \mu(w^*)$.

The quantity $\lim_{w \to \rho} U'(w) \to -\infty$ whenever $\lim_{w \to \rho} U'(w) = \infty$.

$$G_w = \frac{\partial G}{\partial w} = -F(\mu(w), N) + (\rho_L - w) \frac{\partial F}{\partial w} \mu'(w);$$

$F$ is strictly decreasing in $M$, so $G_w \to \infty$ as $w \to 0$ and $G_w = -F(\mu(\rho), N) < 0$ at $w = \rho_L$. For each $N > 0$, $G(w, N)$ attains an interior maximum at $w^* \in (0, \rho_L)$. However $\tilde{w}$ can be zero for $\tilde{N}$ sufficiently large. Finally, $F_N > 0$ for $w < \rho_L$, so that $\max_w \Pi(w, M, N)$ is strictly increasing in $N$. $\square$
Proposition 5. Suppose the conditions of Proposition 4 hold. Let

\[ \varepsilon = U(\rho_H) - U(\rho_L) - (1 - \beta) \frac{\theta}{\pi}. \]

For \( \varepsilon \) sufficiently small and positive, \( w_L > \tilde{w}; \mu(\rho) < \mu(w) < \bar{N} \) whenever \( \bar{N} \) is large enough.

Proof. Let \( w_{IC}(M) = U^{-1}(\alpha - \sigma(1 - \beta q_M)) \) and \( w_{PC}(M) = U^{-1}(\alpha - \sigma'(1 - \beta q_M)) \).

Note that since \( \sigma' = \sigma + \varepsilon/(1 - \beta(1 - \pi)) \) we can write, for each \( M, \)

\[ w_{IC}(M) = w_{PC}(M) + \varepsilon d(M), \]

where \( d \) is positive and bounded for all \( M \leq \bar{N} \). Define

\[ \Pi_{IC}(M, \bar{N}) = (\mu_L - w_{IC}(M))F(M, \bar{N}), \]

\[ \Pi_{PC}(\bar{N}) = (\mu_L - w_{PC}(\bar{N}))F(\bar{N}, \bar{N}), \]

and let \( \Pi^{\ast}(\bar{N}) = \max_{M \leq \bar{N}} \Pi_{IC}(M, \bar{N}), \) and \( \Pi_{\ast}(\bar{N}) = \Pi_{IC}(\bar{N}, \bar{N}). \) By construction, \( \Pi^{\ast}(\bar{N}) \geq \Pi_{\ast}(\bar{N}). \) For \( \bar{N} \) sufficiently large, \( \tilde{w} - 0 \) and \( \partial \Pi_{IC}/\partial M|_{M=\bar{N}} \) is very different from zero, so that \( \Pi^{\ast}(\bar{N}) > \Pi_{\ast}(\bar{N}). \)

The firm chooses \( w > \tilde{w} \) whenever \( \Pi^{\ast}(\bar{N}) > \Pi_{PC}(\bar{N}). \) Note that

\[ \Pi_{\ast}(\bar{N}) = \Pi_{PC}(\bar{N}) - \varepsilon d_{\ast}(\bar{N}), \]

where \( d_{\ast} \) is bounded. At \( \varepsilon = 0, \Pi^{\ast} > \Pi_{PC}. \) The result follows by continuity. \( \square \)

The qualitative properties of interior solutions, with \( w > \tilde{w}, \) are quite different from that of a corner solution with \( w = \tilde{w}. \) At an interior solution \( M < \bar{N}, \) so that not all workers do research (as we noted before, production efficiency requires that \( M = \bar{N}. \)) Also note that an increase in the available labour force can increase wages and decrease average productivity.

We have analysed the behaviour of a firm subject to the constraint that \( N \leq \bar{N}, \) the aggregate supply of labour. It is possible to think of this in two ways; either that the production needs a trained labour force, of which there is a limited supply; alternately, that these are capacity constraints on the size of the firm. If we choose to think of \( \bar{N} \) as a capacity constraint, it alters very little of the previous analysis - other than the issues of market structure. A firm with employment \( \bar{N} \) is capacity-constrained, but not a monopsonist.
6. Innovation ladders

The model of the previous sections will now be generalized in regard to the assumption that there is only one possible thing to learn. We will now allow for countably many possible productivity improvements. The productivity levels are denoted by \( p = (\rho_1, \rho_2, \ldots, \rho_i, \ldots) \). This sequence is assumed to be nondecreasing: \( \rho_{i+1} \geq \rho_i \) for all \( i \). The discovery method is a ladder: if the current state of productivity is \( \rho_i \), then research is assumed to lead to \( \rho_{i+1} \) with probability \( \pi_i \); with probability \( 1 - \pi_i \) we stay in state \( \rho_i \). We will consider a workers cooperative, where every person has claims on their own output. The advantage of forming groups stems from the possibility of learning from others.

Suppose \( M_i \) search at state \( i \) (productivity \( \rho_i \)) and denote by \( V_i \) the value of being at \( i \). In the next period, with probability \( (1 - \pi_i)^{M_i} \) the value will be \( V_i \) and with \( 1 - (1 - \pi_i)^{M_i} \) the group will be in state \( i + 1 \) with value \( V_{i+1} \). As before, let \( q_i \equiv (1 - \pi_i) \). The first problem we examine is that of how many people should do research? This is the socially optimal number, which maximizes the total lifetime utility of the group, and the solution will differ from the number who will actually choose to do research in equilibrium when effort is unobservable. The optimality equation is

\[
V_i = \max_{0 \leq M_i \leq N} \{ NU(\rho_i) + M_i \pi_i [U(\rho_{i+1}) - U(\rho_i)] \\
- \theta M_i + \beta [q_i^{M_i} V_i + (1 - q_i^{M_i}) V_{i+1}] \}
\]

It will be convenient to denote \( U(\rho_{i+1}) - U(\rho_i) \) by \( A(\rho_i) \) and to write \( A V_i \) for \( V_{i+1} - V_i \). The analysis of the equation leads to the conclusion that to maximize total utility the number of people, \( M_i \), who should do research, \( 0 < M_i < N \), should be chosen to satisfy

\[
A(\rho_i) + \beta q_i^{M_i} A V_i \leq \frac{\theta}{\pi_i} \leq A(\rho_i) + \beta q_i^{M_i-1} A V_i.
\]

If we define \( \mu_i \) by

\[
\mu_i = \frac{\ln(\theta - \pi_i A(\rho_i)) - \ln(\beta \pi_i A V_i)}{\ln q_i},
\]

then the determining condition for \( M_i \) is

\[
\mu_i \leq M_i \leq 1 + \mu_i.
\]
This condition is sensible only if \( \theta > \pi_i \Delta(\rho_i) \), but that is the case of interest since, if this condition is violated, all \( N \) should do research.

We turn next to a characterization of the exit condition – when is \( M_i = 0 \) optimal? Once again, assuming \( \pi_i \Delta(\rho_i) < \theta \), research will be stopped at \( \rho_i \) if \( \theta, (\pi_i, \pi_{i+1}, \ldots) \), and \( (\rho_i, \rho_{i+1}, \ldots) \) are such that

\[
\Delta(\rho_i) + \beta \Delta V_i \leq \frac{\theta}{\pi_i}.
\]

In this case, \( \rho_i \) is the maximum achievable productivity.

A somewhat trivial example of such stopping of research would be when \( \rho \) is given by \( \rho = (\rho_L, \rho_H, \rho_H, \ldots) \); the first two states are as in the ‘one-discovery’ model after which every ‘discovery’ has the productivity as \( \rho_H \). After research has been stopped, if by fluke or accident, the system moves to \( \rho_{i+k} \), research could get restarted.

We can now examine the incentive question: when will a worker join \( M_i - 1 \) others in doing research? We replace the optimality equation of the ‘social’ maximization problem with

\[
V_i = \max_{se[0,1]} U(\rho_i) + s[\pi_i \Delta(\rho_i) \theta] \\
+ \beta [q_i(M_i-1+s) V_i + (1 - q_i(M_i-1+s)) V_{i+1}].
\]

and state:

**Lemma 9.** A worker, if self-employed, will choose to do research if and only if

\[
\Delta(\rho_i) + \beta \Delta V_i \geq \frac{\theta}{\pi_i}.
\]

A worker will join \( M_i - 1 \) others in doing research if and only if

\[
\Delta(\rho_i) + \beta q_i(M_i-1) \Delta V_i \geq \frac{\theta}{\pi_i}.
\]

The proof follows Lemmas 1 and 3, and is omitted.

It is easy to confirm that the model developed in this section is a generalization of the ‘one-innovation’ model. Of course, the real question is whether there is something more to be said with the help of this generalization. Stronger conditions are required to show the pattern of stochastic productivity growth.
While a detailed examination of dynamic growth paths is beyond the scope of this paper (and, in any case, requires a computational approach for problems of any generality) we can solve one fairly general model. The probability of success for all innovations is the same ($\pi_i = \pi$ for all $i$); upon success in research, productivity increases by the same amount ($\rho_{i+1} = \rho_i + \alpha$); all workers are risk-neutral (with $U(\rho_i) = \rho_i$) and $A(\rho_i) = \alpha$. It simplifies things to consider the following (equivalent) reformulation with optimality equation

$$V(x) = \max_{0 \leq M \leq N} \left( x + M(\alpha \pi - \theta) + \beta[q^M V(x) + (1 - q^M) V(x + N\alpha)] \right).$$

At the initial date, we have $x = N\rho_o$; research by $M$ leads to $x \pi M$ today (in expected value) and $N\rho_0$ tomorrow with probability $q^M$ or $N\rho_0 + \alpha N$ with probability $(1 - q^M)$, and so on. The important thing to note is that the choice of $M$ is independent of the level of $x$, so that the number of researchers will be constant over time.

When $M$ is constant, we have for all $x$,

$$\Delta V = V(x + \alpha N) - V(x) = \frac{\alpha N}{1 - \beta},$$

i.e., the increments are independent of $M$, different values of $M$ make these increments more or less frequent. Since, for a fixed $M$,

$$V_t(x) = x + M(\alpha \pi - \theta) + \beta[q^M V_{t+1}(x) + (1 - q^M) V_{t+1}(x + N\alpha)],$$

we have the expected output in time $t$,

$$N\rho_0 + M\alpha \pi + t(1 - q^M)\alpha N,$$

so that the output increases by an expected amount $(1 - q^M)\alpha N$ in each period. The total expected utility in time $t$ is

$$N\rho_0 + M\alpha \pi + t(1 - q^M)\alpha N - M\theta.$$ 

It is clearly possible to choose parameters such that self-employed workers do not want to do research ($\alpha < \theta(1 - \beta)/\pi$) and $M > 0$ workers are willing to do research as part of a group larger than one ($\theta(1 - \beta)/\pi < \alpha + (N - 1)\alpha \beta$). This comparison measures the beneficial effects of spillovers, and the per period gain $(1 - q^M)\alpha N$ can then all be attributed to the fact that $N$ can learn from observation. Even if self-employed workers do want to do research, productivity improvements will be much more frequent within groups.
One final thing needs to be considered: the effect of incentive constraints on the choice of $M$. However, as before with the worker cooperative case we take payments to be fixed at $\rho$ (everyone has a claim to their own output). Now a worker will join $M - 1$ in doing research if

$$x + \beta q^{M-1} \frac{x}{1 - \beta} \geq \frac{\theta}{\pi} > x + \beta q^{M} \frac{x}{1 - \beta},$$

which may also be written as

$$x[1 - \beta(1 - q^{M-1})] \geq \frac{\theta}{\pi} (1 - \beta) > x[1 - \beta(1 - q^{M})].$$

The condition for choosing the socially optimal level of $M$ is

$$x[1 - \beta(1 - q^{M-1}N)] \geq \frac{\theta}{\pi} (1 - \beta) > x[1 - \beta(1 - q^{MN})].$$

For $N = 1$ these are, of course, the same conditions, but for $N > 1$ the incentive constraints imply a smaller choice of $M$. The choice of $M$ will be determined here from $\mu \leq M \leq \mu + 1$, where $\mu = \ln((e - WC) (1 - \pi)) - \ln(x\pi)}/\ln q$. The socially optimal choice of $M$ is made from

$$\mu - \frac{\ln N}{\ln q} \leq M \leq \mu + 1 - \frac{\ln N}{\ln q},$$

so that the size is larger by about $-\ln N/\ln q$, which is positive since $\ln q < 0$.

Denote the floor of amount $\lfloor -\ln N/\ln q \rfloor$ by $\gamma$. Now expected output in $t$ is given by $N\rho_0 + Mx_0 + t(1 - q^M)xN$, and (in the socially optimal case) $N\rho_0 + (M + \gamma)x_0 + t(1 - q^M\gamma)xN$, respectively. The difference in expected output at time $t$ is given by

$$\gamma x_0 + txNq^M(1 - q^\gamma).$$

This discrepancy (when $\gamma > 0$) is attributable to the incentive to free ride in the workers' cooperative, to the fact that socially optimal levels of research do not constitute an equilibrium. Note, however, that because of the repeated interaction allowing for richer contracting possibilities would lead to better equilibria.

7. Conclusions and further issues

This model of learning and productivity increase demonstrates the dependence of the production function – individual and aggregate – on the learning
process and the extent to which firms are willing and able to control it. The very possibility of increasing returns to scale in production depends on the wage contracts which the firm offers, and this, in turn, on the nature of competition between firms in the labour market. It is common practice to justify imperfect competition in product markets as a consequence of increasing returns. This analysis suggests that there could very well be an effect going in the other direction stemming from market power in factor markets: firms with monopoly power in the labour market are able to offer wage contracts which result in increasing returns, whereas, in the same economy, price-taking behaviour would ensure constant or decreasing returns. Since the possibility of increasing returns in the aggregate intertemporal production schedule lies at the heart of the endogenous growth paradigm, this suggests intriguing possibilities about the kind of market structures necessary for sustained growth.

The qualitative properties of the production function – most importantly, the nature of returns to scale and the incentives to limit firm size – were shown to depend on the behaviour and objectives of the firm: in particular, whether it is willing and able to manipulate wage contracts to extract maximal profits. Is there any reason to believe that firms will do otherwise? After all, price-taking has little to justify it other than the efficiency properties of competitive equilibria, which are unlikely to hold in this situation. As we have remarked before, price-taking firms can be infinitesimally small, but the whole analysis is close to irrelevant for infinitesimally small firms.

There is a slightly different way to understand the emergence of zero profits under increasing returns to scale, and this is with price-making, or Bertrand equilibria. The standard argument about Bertrand competition with constant returns to scale leading to competitive outcomes continues to hold here. If there are at least two firms and they compete in wage schedules, the only possible outcome is \( w = p \). At any other price, firms make nonzero profits and will either raise wages to bid workers away or cut them. This is because of the form of the profit function: \( \Pi = (\rho L - wL) F(M, N) \).

The paper assumes that research is a binary decision. It should be possible to extend this and allow for variations in research intensity; the probability of discovery could then be made to depend upon the level of research intensity. Generalizations of the search problem in several other directions should also be possible, e.g., when the payoff from research is not clearly known. Learning could involve longer lags, and if imitation also involves some cost, these lags might be endogenously determined. These are avenues we intend to explore in the future.

The assumption of research without recall was useful to set up the problem in a stationary and recursive manner. It is not essential to the qualitative conclusions. With recall, the natural assumption is that \( \pi \) increases with cumulative research. If anything, this reinforces the tendency to free ride on the experience
of others. To undo this, the firm can choose from a richer set of temporal wage profiles.

References